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Form Factors of Branch-Point Twist Fields in Quantum Integrable Models and Entanglement Entropy

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Abstract In this paper we compute the leading correction to the bipartite entanglement entropy at large sub-system size, in integrable quantum field theories with diagonal scattering matrices. We find a remarkably universal result, depending only on the particle spectrum of the theory and not on the details of the scattering matrix. We employ the "replica trick" whereby the entropy is obtained as the derivative with respect to n of the trace of the nth power of the reduced density matrix of the sub-system, evaluated at $n = 1$. The main novelty of our work is the introduction of a particular type of twist fields in quantum field theory that are naturally related to branch points in an n-sheeted Riemann surface. Their two-point function directly gives the scaling limit of the trace of the nth power of the reduced density matrix. Taking advantage of integrability, we use the expansion of this two-point function in terms of form factors of the twist fields, in order to evaluate it at large distances in the two-particle approximation. Although this is a well-known technique, the new geometry of the problem implies a modification of the form factor equations satisfied by standard local fields of integrable quantum field theory. We derive the new form factor equations and provide solutions, which we specialize both to the Ising and sinh-Gordon models.

Keywords Integrable quantum field theory · Entanglement entropy · Form factors · Twist fields

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1 Introduction

Quantum field theory (QFT) has proven to be one of the most successful theories of the physical world. Its main objects are correlation functions of local fields: they describe quantum correlations between separated local observables and provide all physical information that can be extracted from a model of QFT. In general, correlation functions can be computed only perturbatively, giving expressions that apply only in the large-energy region. In contrast, some two-dimensional QFTs (one space and one time dimension) have the property of integrability, which means that correlation functions are accessible non-perturbatively. In particular, their low-energy or large-distance behavior is accessible in exact form in many cases, from exact expressions for so-called form factors of local fields [1, 2]. This fact has triggered an enormous amount of work in computing form factors in a multitude of models of integrable QFT (IQFT), some of which has found applications to low-dimensional condensed matter systems [3].

The main characteristic of a quantum system, as opposed to a classical one, is the existence of entanglement: performing a local measurement may instantaneously affect local measurements far away. This property is essential to the field of quantum computation and teleportation. At the theoretical level, there has been considerable interest in formulating measures of quantum entanglement [4–8] and applying them to extended quantum systems with many degrees of freedom, such as quantum spin chains [9–17]. One of these measures is entanglement entropy [4]. Consider a quantum system, with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, in a pure state $|\psi\rangle$. The bipartite entanglement entropy S_A is the von Neumann entropy associated to the reduced density matrix of the subsystem A, defined as

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|), \tag{1.1}$$

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}(\rho_A \log(\rho_A)). \tag{1.2}$$

A recent application of QFT has been to the calculation of entanglement entropy for the case of one-dimensional systems [18, 19], extending primarily on the work [20], where $\mathcal{H}_A$ is spanned by the degrees of freedom in some interval A (or set of intervals) of the real line, and B is its complement. The authors evaluated the bipartite entanglement entropy in quantum systems at criticality, using techniques of conformal field theory (CFT), as well as the leading large-distance limit in a massive QFT. For example, when $|\psi\rangle$ is the ground state and A is an interval of length r in an infinite system, they found

$$S_A \sim \begin{cases} \dfrac{c}{3}\log(r/\epsilon), & \epsilon \ll r \ll m^{-1}, \\ -\dfrac{c}{3}\log(\epsilon m), & r \gg m^{-1}, \end{cases} \tag{1.3}$$

where m^{-1} is a correlation length of the QFT, ϵ is some short-distance cutoff, and c is the central charge of the CFT.

In this paper, we will develop a framework for the computation of entanglement entropy in massive IQFT using factorized scattering techniques. Our main result is the form of the first sub-leading corrections to the entanglement entropy at large distances. Remarkably, we show that the leading r-dependent correction is independent of the precise details of the S-matrix, being entirely determined by the spectrum of masses of the IQFT whenever the scattering between particles does not involve backscattering. That is,

$$S_A = -\frac{c}{3}\log(\epsilon m_1) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O(e^{-3rm_1}), \tag{1.4}$$

where α labels the particle types, m_α are the associated masses (with m_1 the smallest one), U is a model-dependent constant, and $K_0(z)$ is the modified Bessel function. For free theories, this result was previously obtained by a different approach [21, 22]. The constant U depends on the definition of ϵ, but it can be fixed by requiring that no constant correction terms occur, for instance, in the upper expression in (1.3). With such a definition, we evaluated U in the quantum Ising model, with the result:

$$U_{\text{Ising}} = \frac{1}{6}\log 2 - \int_0^\infty \frac{dt}{2t}\left(\frac{t\cosh t}{\sinh^3 t} - \frac{1}{\sinh^2 t} - \frac{e^{-2t}}{3}\right) = -0.131984\ldots, \tag{1.5}$$

which, as we will explain in Sect. 5, is in very good agreement with existing numerical results [11]. This constant could in principle also be recovered by numerically integrating the Painlevé V equation according to the results obtained in [21].

The main novelty of our approach is the introduction of a certain type of twist field $\mathcal{T}$ in IQFT, whose two-point function is directly related to the entanglement entropy. The initial idea [18] is to evaluate the entanglement entropy by a "replica trick" from the partition function on a multi-sheeted Riemann surface. The field we introduce naturally arises as a local field in an n-copy version of a given model of IQFT, and implements branch points so that its correlation functions are partition functions on multi-sheeted Riemann surfaces. The large-distance corrections to the bipartite entanglement entropy is derived from a large-distance expansion of the two-point function of $\mathcal{T}$, obtained by evaluating its form factors. Due to the new geometry of the problem, the form factor equations differ from their usual form derived thirty years ago [1, 2]. The operation of evaluating the entanglement entropy from the result of the form factor expansion involves a subtle analytic continuation in n. We provide a general derivation of the form factor equations and of the entanglement entropy in the context of diagonal factorised scattering theory, and we have studied the Ising and sinh-Gordon cases in detail.

The paper is organised as follows: in Sect. 2, we discuss partition functions on multi-sheeted spaces and their relation to entanglement entropy in general terms. In Sect. 3, we develop the form factor program for twist fields, and provide a general formula for the two-particle form factors of theories with a single particle spectrum and no bound states, which we specialise to both the Ising and the sinh-Gordon cases. In Sect. 4 we check our previous formulae for consistency by computing the ultraviolet conformal dimension of the twist field in the two-particle approximation both for the Ising and sinh-Gordon model. In Sect. 5 we compute the two-point function of the twist field in the two-particle approximation and derive from that our general formula for the bipartite entanglement entropy for theories with a single particle spectrum and no bound states. In Sect. 6 we extend the previous results to diagonal theories with many particles and bound states. In Sect. 7 we present our conclusions and point out some open problems. Finally we provide four appendices: in Appendix 1 we give an alternative derivation of the form factors of the twist field in the n-copy sinh-Gordon theory using the method of angular quantization; in Appendix 2 we compute the vacuum expectation value of the twist field in the n-copy Ising model and derive the value (1.5) from it; in Appendix 3 we present the details of the analytic continuation necessary for the computation of the entropy and in Appendix 4 we provide the $n \to \infty$ limit of the two-point function of the twist field in the sinh-Gordon model in the two-particle and saddle-point approximation.

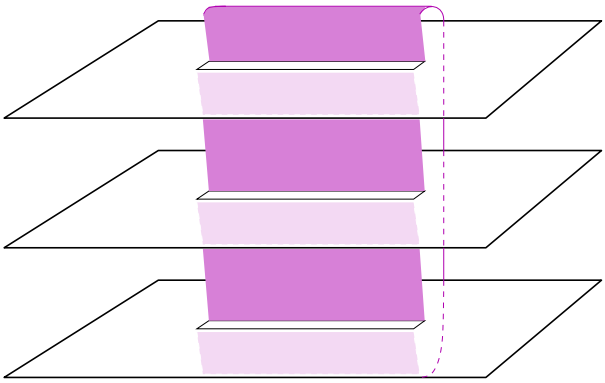

Fig. 1 (Color online) A representation of the Riemann surface $\mathcal{M}_{3,a_1,a_2}$

2 Partition Functions on Multi-Sheeted Spaces and Entanglement Entropy

2.1 Partition Functions in QFT on Multi-Sheeted Spaces

The partition function of a model of two-dimensional QFT with local lagrangian density $\mathcal{L}[\varphi](x, y)$ on a (euclidean-signature) Riemann surface $\mathcal{R}$ is formally obtained by the path integral

$$Z[\mathcal{L}, \mathcal{R}] = \int [d\varphi]_{\mathcal{R}} \exp\left[- \int_{\mathcal{R}} dx dy \, \mathcal{L}[\varphi](x, y) \right], \tag{2.1}$$

where $[d\varphi]_{\mathcal{R}}$ is an infinite measure on the set of configurations of some field φ living on the Riemann surface $\mathcal{R}$ and on which the lagrangian density depends in a local way. Consider Riemann surfaces with curvature zero everywhere except at a finite number of points. Since the lagrangian density does not depend explicitly on the Riemann surface as a consequence of its locality, it is expected that this partition function can be expressed as an object calculated from a model on $\mathbb{R}^2$, where the structure of the Riemann surface is implemented through appropriate boundary conditions around the points with non-zero curvature. Consider for instance the simple Riemann surface $\mathcal{M}_{n,a_1,a_2}$ composed of n sheets sequencially joined to each other on the segment $x \in [a_1, a_2]$, $y = 0$ (see Fig. 1 representing the case $n = 3$). We would expect that the associated partition function involves certain "fields"[1] at $(x, y) = (a_1, 0)$ and $(x, y) = (a_2, 0)$.

The expression (2.1) for the partition function essentially defines these fields (that is, it gives their correlation functions, up to a normalisation independent of their positions). But in the model on $\mathbb{R}^2$, this definition makes them non-local. Locality of a field (used here in its most fundamental sense) means that as an observable in the quantum theory, it is quantum mechanically independent of the energy density at space-like distances. In the associated euclidean field theory, this means that correlation functions involving this field and the energy density are, as functions of the position of the energy density, defined on $\mathbb{R}^2$ (and smooth except at the positions of the fields). The energy density is simply obtained from the lagrangian density, hence it is clear that fields defined by (2.1) in the model on

[1] Here, the term "field" is taken in its most general QFT sense: it is an object of which correlation functions—multi-linear maps—can be evaluated, and which depends on a position in space—parameters x, y that transform like coordinates under translation symmetries.

$\mathbb{R}^2$ are not local. Locality is at the basis of most of the results in integrable QFT, so it is important to recover it.

The idea is simply to consider a larger model: a model formed by n independent copies of the original model, where n is the number of Riemann sheets necessary to describe the Riemann surface by coordinates on $\mathbb{R}^2$. Let us take again the simple example of $\mathcal{M}_{n,a_1,a_2}$. We re-write (2.1) as

$$Z[\mathcal{L},\mathcal{M}_{n,a_1,a_2}]=\int_{\mathcal{C}(a_1;a_2)}[d\varphi_1\cdots d\varphi_n]_{\mathbb{R}^2}\exp\left[-\int_{\mathbb{R}^2}dxdy\,(\mathcal{L}[\varphi_1](x,y)+\cdots+\mathcal{L}[\varphi_n](x,y))\right], \tag{2.2}$$

where $\mathcal{C}(a_1,a_2)$ are *conditions* on the fields $\varphi_1,\ldots,\varphi_n$ restricting the path integral:

$$\mathcal{C}(a_1;a_2):\varphi_i(x,0^+)=\varphi_{i+1}(x,0^-),\quad x\in[a_1,a_2],\ i=1,\ldots,n, \tag{2.3}$$

where we identify $n+i\equiv i$. The lagrangian density of the multi-copy model is

$$\mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x,y)=\mathcal{L}[\varphi_1](x,y)+\cdots+\mathcal{L}[\varphi_n](x,y),$$

so that the energy density in that model is the sum of the energy densities of the n individual copies. Hence the expression (2.2) does indeed define local fields at $(a_1,0)$ and $(a_2,0)$ in the multi-copy model, since this sum is the same on both sides of the segment $x\in[a_1,a_2]$, $y=0$ according to the conditions $\mathcal{C}(a_1,a_2)$.

The local fields defined in (2.2) are examples of "twist fields". Twist fields exist in a QFT model whenever there is a global internal symmetry σ (a symmetry that acts the same way everywhere in space, and that does not change the positions of fields): $\int_{\mathbb{R}^2}dxdy\,\mathcal{L}[\sigma\varphi](x,y)=\int_{\mathbb{R}^2}dxdy\,\mathcal{L}[\varphi](x,y)$. Their correlation functions can be formally defined through the path integral:

$$\langle\mathcal{T}_\sigma(a,b)\cdots\rangle_{\mathcal{L},\mathbb{R}^2}\propto\int_{\mathcal{C}_\sigma(0,0)}[d\varphi]_{\mathbb{R}^2}\exp\left[-\int_{\mathbb{R}^2}dxdy\,\mathcal{L}[\varphi](x,y)\right]\cdots, \tag{2.4}$$

where $\cdots$ represent insertions of other local fields at different positions and the path integral conditions are

$$\mathcal{C}_\sigma(a,b):\varphi(x,b^+)=\sigma\varphi(x,b^-),\quad x\in[a,\infty). \tag{2.5}$$

The proportionality constant is an infinite constant that is independent of the position (a,b) and of those of the other local fields inserted. The fact that σ is a symmetry ensures that $\mathcal{T}_\sigma$ is local. Also, it insures that the result is in fact independent of the shape of the cut in the conditions $\mathcal{C}_\sigma$, up to symmetry transformations of the other local fields inserted. A consequence of this definition is that correlation functions $\langle\mathcal{T}_\sigma(a,b)\mathcal{O}(x,y)\cdots\rangle_{\mathcal{L},\mathbb{R}^2}$ with some local fields $\mathcal{O}(x,y)$ are defined, as functions of x,y (smooth except at positions of other local fields), on a multi-sheeted covering of $\mathbb{R}^2$ with a branch point at (a,b), whenever $\sigma\mathcal{O}\neq\mathcal{O}$. They have the property that a clockwise turn around (a,b) is equivalent to the replacement $\mathcal{O}\mapsto\sigma\mathcal{O}$ in the correlation function. If $\sigma\mathcal{O}\neq\mathcal{O}$, then $\mathcal{O}$ is said to be "semi-local" with respect to $\mathcal{T}_\sigma$. This property, along with the condition that $\mathcal{T}_\sigma$ has the lowest scaling dimension and be invariant under all symmetries of the model that commute with σ (that is, that it be a primary field in the language of conformal field theory), is expected to uniquely fix the field $\mathcal{T}_\sigma$, and constitute a more fundamental definition than the path integral above, as it does not require the existence of a lagrangian density. We will take this point of view in the following, but we will continue to denote a model of QFT by $\mathcal{L}$ and its n-copy tensor product by $\mathcal{L}^{(n)}$.

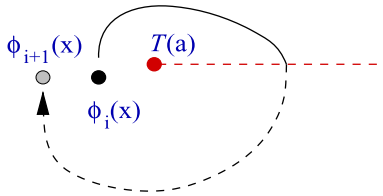

Fig. 2 (Color online) The effect of $\mathcal{T}$ on other local fields

In the model with lagrangian $\mathcal{L}^{(n)}$, there is a symmetry under exchange of the copies. The twist fields defined by (2.2), which we call *branch-point twist fields*, are twist fields associated to the two opposite cyclic permutation symmetries $i \mapsto i+1$ and $i+1 \mapsto i$ ($i = 1, \ldots, n, n+1 \equiv 1$). We will denote them simply by $\mathcal{T}$ and $\tilde{\mathcal{T}}$, respectively:

$$\begin{aligned} \mathcal{T} &= \mathcal{T}_\sigma, \qquad \sigma : i \mapsto i+1 \mod n, \\ \tilde{\mathcal{T}} &= \mathcal{T}_{\sigma^{-1}}, \quad \sigma^{-1} : i+1 \mapsto i \mod n \end{aligned}$$

(see Fig. 2 for the case $\mathcal{T}$). More precisely, we have

$$Z[\mathcal{L}, \mathcal{M}_{n,a_1,a_2}] \propto \langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}. \tag{2.6}$$

This can be seen by observing that for $x \in [a_1, a_2]$, consecutive copies are connected through $y = 0$ due to the presence of $\mathcal{T}(a_1, 0)$, whereas for $x > a_2$, copies are connected to themselves through $y = 0$ because the conditions arising from the definition of $\mathcal{T}(a_1, 0)$ and $\tilde{\mathcal{T}}(a_2, 0)$ cancel each other.

More generally, the identification holds for correlation functions in the model $\mathcal{L}$ on $\mathcal{M}_{n,a_1,a_2}$, this time with an equality sign:

$$\langle \mathcal{O}(x, y; \text{sheet } i)\cdots\rangle_{\mathcal{L},\mathcal{M}_{n,a_1,a_2}} = \frac{\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\mathcal{O}_i(x,y)\cdots\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}, \tag{2.7}$$

where $\mathcal{O}_i$ is the field in the model $\mathcal{L}^{(n)}$ coming from the ith copy of $\mathcal{L}$.

The generalisation to Riemann surfaces with more branch points is straightforward, but will not be needed here.

The conformal dimension of branch-point twist fields was calculated[2] in [18]. Consider the model $\mathcal{L}$ to be a conformal field theory (CFT). Then also $\mathcal{L}^{(n)}$ is a CFT. There are n fields $T_j(z)$ in $\mathcal{L}^{(n)}$ that correspond to the stress-energy tensors of the n copies of $\mathcal{L}$, and in particular the sum $T^{(n)}(z) = \sum_{j=1}^n T_j(z)$ is the stress-energy tensor of $\mathcal{L}^{(n)}$. The central charge of $\mathcal{L}^{(n)}$ is nc, if c is that of $\mathcal{L}$.

Consider the stress-energy tensor $T(w)$ in $\mathcal{L}$. We can evaluate the one-point function $\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,a_1,a_2}}$ by making a conformal transformation from z in $\mathbb{R}^2$ to w in $\mathcal{M}_{n,a_1,a_2}$ (here z and w are complex coordinates, with for instance $z = x + iy$) given by

$$z = \left(\frac{w-a_1}{w-a_2}\right)^{\frac{1}{n}}.$$

[2] In fact, in the paper [18] branch-point twist fields in the multi-copy model were not introduced explicitly, as they are not essential for the evaluation of partition functions in CFT. Only the non-local fields discussed above were alluded to, but the method to evaluate the scaling dimension is the same.

We have

$$\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,a_1,a_2}} = \left(\frac{\partial z}{\partial w}\right)^2 \langle T(z)\rangle_{\mathcal{L},\mathbb{R}^2} + \frac{c}{12}\{z,w\},$$

where the Schwarzian derivative is

$$\{z,w\} = \frac{z'''z' - (3/2)(z'')^2}{(z')^2}.$$

Using $\langle T(z)\rangle_{\mathcal{L},\mathbb{R}^2} = 0$, we obtain

$$\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,a_1,a_2}} = \frac{c(n^2-1)}{24n^2}\frac{(a_1-a_2)^2}{(w-a_1)^2(w-a_2)^2}.$$

Since, by (2.7), this is equal to $\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)T_j(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}/\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}$ for all j, we can evaluate the correlation function involving the stress-energy tensor of $\mathcal{L}^{(n)}$ by multiplying by n:

$$\frac{\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(a_1,0)\mathcal{T}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}} = \frac{c(n^2-1)}{24n}\frac{(a_1-a_2)^2}{(w-a_1)^2(w-a_2)^2}.$$

From the usual CFT formula for insertion of a stress-energy tensor

$$\begin{aligned}&\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}\\ &\quad= \left(\frac{1}{w-a_1}\frac{\partial}{\partial a_1} + \frac{h_1}{(w-a_1)^2} + \frac{1}{w-a_2}\frac{\partial}{\partial a_2} + \frac{h_2}{(w-a_2)^2}\right)\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}\end{aligned}$$

we identify the scaling dimension of the primary fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ (they have the same scaling dimension) using $\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} = |a_1-a_2|^{-2d_n}$:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right). \tag{2.8}$$

It may happen that many fields with the main property of branch-point twist fields exist, with different dimensions. However, the dimension (2.8) should be the lowest possible dimension. Hence, a field with the main properties of branch-point twist field, with this dimension, and invariant under all symmetries of the theory should be unique.

2.2 Entanglement Entropy

Partition functions on Riemann surfaces with branch points can be used in order to evaluate the entanglement entropy; this works when A consists of one and also of more than one interval, as was explained in [18]. Consider a (finite or infinite) one-dimensional quantum system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ is the space of local degrees of freedom in some interval (or set of intervals) A. Consider also the ground state $|\psi\rangle$ of this quantum system. The bipartite entanglement entropy S_A is defined as follows. We first define the induced density matrix ρ_A as

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|) \tag{2.9}$$

and then we calculate the von Neumann entropy associated to this density matrix:

$$S_A = -\operatorname{Tr}_{\mathcal{H}_A}(\rho_A \log(\rho_A)). \tag{2.10}$$

This has the interpretation of counting the number of fully entangled "links" between the regions A and $\bar{A}$ as encoded into the ground state $|\psi\rangle$. In agreement with this interpretation, it has the symmetry property $S_A = S_{\bar{A}}$, and in a quantum model with local interaction, it is expected to saturate to a finite value when both regions A and $\bar{A}$ are much larger than the correlation length.

The main idea in order to evaluate the entanglement entropy in the scaling limit of a quantum model (here, we will consider models on infinite space only) is to use the "replica trick" [18]. That is, we evaluate

$$\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n, \tag{2.11}$$

then take the limit $n \to 1$ of the derivative with respect to n, using the identity $\rho_A \log \rho_A = \lim_{n\to 1} \frac{\partial}{\partial n} \rho_A^n$:

$$S_A = -\lim_{n\to 1} \frac{d}{dn} \operatorname{Tr}_{\mathcal{H}_A} \rho_A^n. \tag{2.12}$$

This formula requires that we evaluate the trace with real positive n, but the trick is to evaluate $\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n$ with positive integer n, then to take the appropriate "analytic continuation" (it is unique if we assume a certain asymptotic behaviour as $n \to \infty$—we will discuss this in Sect. 5). Considering positive integer n is useful, because in the scaling limit, denoting the QFT model associated to the region near the critical point by $\mathcal{L}$, we have, taking A to consist of only one interval,

$$\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n \to Z[\mathcal{L}; \mathcal{M}_{n,a_1,a_2}], \tag{2.13}$$

where a_1 and a_2 are the dimensionful end-points of the region A (the scaling limit is taken with the length of the region A in proportion to the correlation length, which is then sent to infinity). As we saw above, this can be computed as a two-point correlation function of local fields in $\mathcal{L}^{(n)}$ using (2.6). More precisely, with m a mass scale of the QFT and ϵ some dimensionful distance of the order of the site spacing, $m\epsilon$ being in inverse proportion to the dimensionless correlation length, we have

$$\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n \sim \mathcal{Z}_n \epsilon^{2d_n} \langle \mathcal{T}(a_1, 0)\tilde{\mathcal{T}}(a_2, 0)\rangle_{\mathcal{L}^{(n)}, \mathbb{R}^2} \tag{2.14}$$

with an n-dependent non-universal normalisation constant $\mathcal{Z}_n$ (with $\mathcal{Z}_1 = 1$), and where d_n is the scaling dimension (2.8). For later convenience, ϵ is chosen in such a way that $d\mathcal{Z}_n/dn = 1$ at $n = 1$. Note that the expression above is dimensionless, since the operators $\mathcal{T}$ and $\tilde{\mathcal{T}}$ both have dimension d_n (in particular, their individual vacuum expectation value in the QFT is proportional to m^{d_n}).

Similarly, for regions composed of many disconnected components, $\operatorname{Tr}_{\mathcal{H}_A} \rho_A^n$ is identified with partition functions on Riemann surfaces with many branch points, as explained in [18], but we will not consider this case here.

3 The Form Factor Program for Branch-Point Twist Fields

We now turn to the description of QFT on Minkowski space-time in terms of its Hilbert space of asymptotic relativistic particles. In the context of $1 + 1$-dimensional IQFT, form

factors are defined as tensor valued functions representing matrix elements of some local operator $\mathcal{O}(x)$ located at the origin $x = 0$ between a multi-particle *in*-state and the vacuum:

$$F_k^{\mathcal{O}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle^{\text{in}}_{\mu_1,\ldots,\mu_k}. \tag{3.1}$$

Here $|0\rangle$ represents the vacuum state and $|\theta_1,\ldots,\theta_k\rangle^{\text{in}}_{\mu_1,\ldots,\mu_k}$ the physical "in" asymptotic states of massive QFT. They carry indices μ_i, which are quantum numbers characterizing the various particle species, and depend on the real parameters θ_i, which are called rapidities. The form factors are defined for all rapidities by analytically continuing from some ordering of the rapidities; a fixed ordering provides a complete basis of states.

The main characteristics of massive integrable models of QFT are that the number of particles and their momenta set are conserved under scattering, and that the scattering matrix factorises into products of two-particle scattering matrices, which are the solutions of a set of consistency equations and analytic properties. These consistency equations and analytic properties are often strong enough to completely fix the scattering matrix in integrable models. Similarly, the form factors are fixed by a set of equations and analytic properties depending on the two-particle scattering matrix (or S-matrix).

In this section we want to show how the standard form factor equations for $1+1$-dimensional IQFTs must be modified for the branch-point twist fields. Let us consider an integrable model consisting of n copies of a known integrable theory possessing a single particle spectrum and no bound states (such as the Ising and sinh-Gordon models). We have therefore n particles, which we will denote by indices $1,\ldots,n$. The S-matrix between particles i and j with rapidities θ_i and θ_j will be denoted by $S_{ij}(\theta_i-\theta_j)$ (that it depends on the rapidity difference is a consequence of relativistic invariance). Particles of different copies do not interact with each other, so that the S-matrix of the model will be of the form

$$S_{ii}(\theta) = S(\theta), \quad \forall i = 1,\ldots,n, \tag{3.2}$$

$$S_{ij}(\theta) = 1, \quad \forall i,j = 1,\ldots,n \text{ and } i \neq j, \tag{3.3}$$

where $S(\theta)$ is the S-matrix of the single-copy integrable QFT. As explained above, as a consequence of the symmetry of the model, a twist field $\mathcal{T}$ must exist such that if $\Psi_1,\ldots,\Psi_n$ are the fields associated to the fundamental particles of each copy of the original model, then the equal time ($x^0 = y^0$) exchange relations between $\mathcal{T}$ and $\Psi_1,\ldots,\Psi_n$ can be written in the following form[3]

$$\begin{aligned} &\Psi_i(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_{i+1}(y), \quad x^1 > y^1, \\ &\Psi_i(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_i(y), \quad x^1 < y^1, \end{aligned} \tag{3.4}$$

for $i = 1,\ldots,n$ and where we identify the indices $n+i \equiv i$. The relation with the previous section is obtained by recalling that in going from the Hilbert space description to the path integral description, the order of operators is translated into time-ordering (or y-ordering in euclidean space), and that left-most operators are later in time. It is well known that such exchange relations play an important role in the derivation of the consistency equations for the form factors of the operator $\mathcal{T}$. Generalising the standard arguments to the exchange

[3]Here we employ the standard notation in Minkowski space-time: x^ν with $\nu = 0, 1$, with x^0 being the time coordinate and x^1 being the position coordinate.

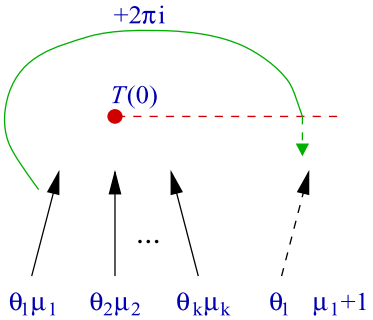

Fig. 3 (Color online) A pictorial representation of the effect of adding $2\pi i$ to rapidity θ_1 in form factors of the twist field $\mathcal{T}$

relation (3.4), the form factor axioms are

$$\begin{aligned}
&F_k^{\mathcal{T}|\cdots\mu_i\mu_{i+1}\cdots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S_{\mu_i\mu_{i+1}}(\theta_{i\,i+1})F_k^{\mathcal{T}|\cdots\mu_{i+1}\mu_i\cdots}(\ldots,\theta_{i+1},\theta_i,\ldots),\\
&F_k^{\mathcal{T}|\mu_1\mu_2\cdots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2\cdots\mu_n\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1),\\
&-i\mathrm{Res}_{\bar{\theta}_0=\theta_0}F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\cdots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k),\\
&-i\mathrm{Res}_{\bar{\theta}_0=\theta_0}F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\cdots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = -\prod_{i=1}^{k}S_{\mu\mu_i}(\theta_{0i})F_k^{\mathcal{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k).
\end{aligned} \tag{3.5}$$

Here $\theta_{ij}=\theta_i-\theta_j$ and the first axiom is in fact the same as for local fields. In the second equation, the crossing or locality relation, we introduced the symbols $\hat{\mu}_i=\mu_i+1$. As compared to the usual form factor equations, it is altered by the nature of the exchange relation and it now relates form factors associated to different particle sets. Finally, the last two equations generalise the standard kinematic residue equation to branch-point twist fields. Once more, the exchange relations (3.4) are responsible for the splitting into two equations. Here, for later convenience, we wrote the equations in their general form valid also for many-particle models, where $\bar{\mu}$ represents the anti-particle associated to μ. In the present case, the integrable model we started with has just one particle (so that μ labels the copies) and therefore each particle is its own anti-particle. Since we are dealing with theories without bound states, these are in fact all the equations which one needs to solve. Pictorial explanations of the second and of the last two equations are given, respectively, in Figs. 3 and 4.

The other field $\tilde{\mathcal{T}}$ introduced in Sect. 2 is a twist field with similar properties as $\mathcal{T}$ but whose exchange relations with the fundamental fields of the theory are given by

$$\begin{aligned}
\Psi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Psi_{i-1}(y), \quad x^1>y^1,\\
\Psi_i(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\Psi_i(y), \quad x^1<y^1,
\end{aligned} \tag{3.6}$$

instead of (3.4). This implies that on the Hilbert space, we have

$$\tilde{\mathcal{T}}=\mathcal{T}^\dagger. \tag{3.7}$$

In order to fully define the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$, we need to fix their normalisation, which does not follow from the form factor equations. We will adopt the usual CFT normalisation:

$$\langle\mathcal{T}(x)\tilde{\mathcal{T}}(0)\rangle \sim r^{-2d_n} \quad \text{as } r\to 0. \tag{3.8}$$

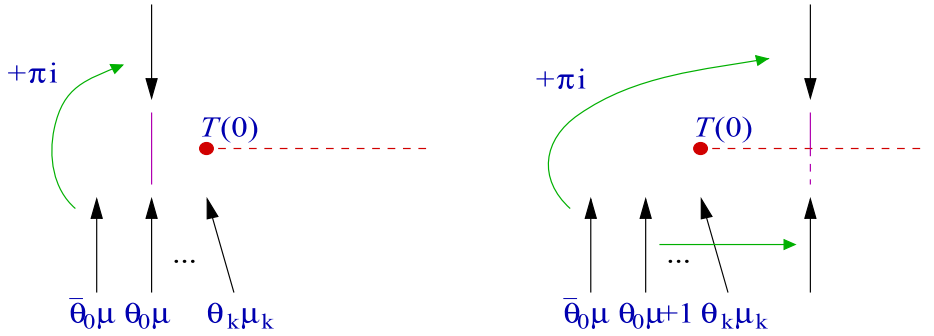

Fig. 4 (Color online) The kinematic poles come from the structure of the wave function far from the local fields, at positive and negative infinity. Adding $i\pi$ to rapidity θ_1 puts the particle in the "out" region. With a particle in that region, there are delta-functions representing particles in the "in" region going through without interacting with the local fields. Those occur from the e^{ipx} form of the wave function at positive and negative infinity. But if the coefficients at both limits are different, $S_- e^{ipx}$ and $S_+ e^{ipx}$ with $S_- \neq S_+$, then there are also poles in addition to these delta-functions. Only these poles are seen in the analytic continuation $\theta_1 \mapsto \theta_1 + i\pi$. Different coefficients come from the semi-locality of the twist field and the non-free scattering matrix, as represented here

Here and below, r denotes the space-like separation $\sqrt{(x^1)^2 - (x^0)^2}$, and the two-point function just depends on it thanks to relativistic invariance and spinless-ness of the fields involved.

3.1 Two-Particle Form Factors

As usual in this context, we define the minimal form factors $F^{\mathcal{T}|jk}_{\min}(\theta, n)$ to be solutions of the first two equations in (3.5) for $k = 2$ without poles in the physical sheet $\mathrm{Im}(\theta) \in [0, \pi]$. That is,

$$F^{\mathcal{T}|kj}_{\min}(\theta, n) = F^{\mathcal{T}|jk}_{\min}(-\theta, n) S_{kj}(\theta) = F^{\mathcal{T}|j\,k+1}_{\min}(2\pi i - \theta, n), \quad \forall j, k \tag{3.9}$$

where the S-matrix is given by (3.2). Repeated use of the above equations leads to the following constraints:

$$F^{\mathcal{T}|i\,i+k}_{\min}(\theta, n) = F^{\mathcal{T}|j\,j+k}_{\min}(\theta, n), \quad \forall i, j, k, \tag{3.10}$$

$$F^{\mathcal{T}|1j}_{\min}(\theta, n) = F^{\mathcal{T}|11}_{\min}(2\pi(j-1)i - \theta, n), \quad \forall j \neq 1. \tag{3.11}$$

These equations show that computing just the form factor $F^{\mathcal{T}|11}_{\min}(\theta, n)$ is enough to determine all minimal form factors of the theory. A consequence of these equations is that this minimal form factor must have no poles in the extended strip $\mathrm{Im}(\theta) \in [0, 2\pi n]$. From the equations above it is easy to deduce

$$F^{\mathcal{T}|11}_{\min}(\theta, n) = F^{\mathcal{T}|11}_{\min}(-\theta, n) S(\theta) = F^{\mathcal{T}|11}_{\min}(-\theta + 2\pi n i, n). \tag{3.12}$$

In order to develop a systematic procedure to solve these equations it is useful to recall that, for a standard local operator the minimal form factor equations take the form

$$f_{11}(\theta) = f_{11}(-\theta) S(n\theta) = f_{11}(-\theta + 2\pi i), \tag{3.13}$$

provided that the S-matrix of the theory is given by $S(n\theta)$. Thus given a solution to the previous equation, the function $F_{\min}^{\mathcal{T}|11}(\theta,n)=f_{11}(\theta/n)$ is automatically a solution of (3.12).

In the context of integrable models, a systematic way of solving such type of equations has been developed whereby, given an integral representation for $S(\theta)$, an integral representation of $f_{11}(\theta)$ can be readily obtained [1]. For diagonal theories, the integral representation of the S-matrix takes the form

$$S(\theta)=\exp\left[\int_0^\infty \frac{dt}{t}g(t)\sinh\left(\frac{t\theta}{i\pi}\right)\right], \tag{3.14}$$

where $g(\theta)$ is a function which depends of the theory under consideration. A trivial consequence of the previous equation is

$$S(n\theta)=\exp\left[\int_0^\infty \frac{dt}{t}g(t/n)\sinh\left(\frac{t\theta}{i\pi}\right)\right], \tag{3.15}$$

and from here, it is easy to show that

$$f_{11}(\theta)=\mathcal{N}\exp\left[\int_0^\infty \frac{dt}{t\sinh(nt)}g(t)\sin^2\left(\frac{itn}{2}\left(1+\frac{i\theta}{\pi}\right)\right)\right], \tag{3.16}$$

where $\mathcal{N}$ is a normalization constant. Therefore, the desired solution is

$$F_{\min}^{\mathcal{T}|11}(\theta,n)=f_{11}(\theta/n)=\mathcal{N}\exp\left[\int_0^\infty \frac{dt}{t\sinh(nt)}g(t)\sin^2\left(\frac{it}{2}\left(n+\frac{i\theta}{\pi}\right)\right)\right]. \tag{3.17}$$

So far in this section we have computed the minimal form factors. However what we ultimately need are the full two-particle form factors. These are solutions of (3.9) which include poles in the extended physical strip mentioned before. Their pole structure is determined by the kinematical residue equations together with (3.9). According to these equations the form factor $F_2^{\mathcal{T}|11}(\theta,n)$ has two poles in the extended physical sheet at $\theta=i\pi$ and $\theta=i\pi(2n-1)$. It is not difficult to show that a solution of (3.9) which is consistent with the above pole structure is given by

$$F_2^{\mathcal{T}|jk}(\theta)=\frac{\langle\mathcal{T}\rangle\sin(\frac{\pi}{n})}{2n\sinh(\frac{i\pi(2(j-k)-1)+\theta}{2n})\sinh(\frac{i\pi(2(k-j)-1)-\theta}{2n})}\frac{F_{\min}^{\mathcal{T}|jk}(\theta,n)}{F_{\min}^{\mathcal{T}|jk}(i\pi,n)}, \tag{3.18}$$

where the normalization has been chosen so that the kinematical residue equation gives

$$F_0^{\mathcal{T}}=\langle\mathcal{T}\rangle. \tag{3.19}$$

In addition, the constant factor $\sinh(\pi/n)$ guarantees that all form factors vanish for $n=1$ as expected, since in that case the field $\mathcal{T}$ can be identified with the identity. The structure of two-particle form factors is depicted in Fig. 5.

For the field $\tilde{\mathcal{T}}$, the exchange relations imply that form factors of the field $\tilde{\mathcal{T}}$ are equal to those of the field $\mathcal{T}$ up to the transformation $i\to n-i$ for each particle i. At the level of the two particle form factors, this means that

$$F_2^{\mathcal{T}|ij}(\theta,n)=F_2^{\tilde{\mathcal{T}}|(n-i)(n-j)}(\theta,n). \tag{3.20}$$

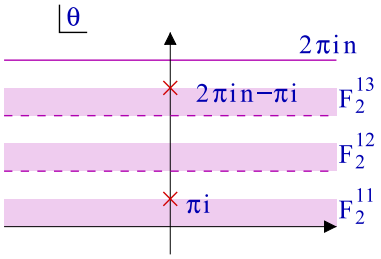

Fig. 5 (Color online) The structure of the function $F_2^{\mathcal{T}|11}(\theta)$ in the extended sheet $\mathrm{Im}(\theta) \in [0, 2\pi n]$, in the case $n = 3$. Crosses indicate the positions of the kinematic singularities. Shaded regions represent the physical sheets of the form factors $F_2^{\mathcal{T}|11}(\theta)$, $F_2^{\mathcal{T}|12}(\theta)$ and $F_2^{\mathcal{T}|13}(\theta)$

This property can be combined with (3.10, 3.11) to show that

$$F_2^{\tilde{\mathcal{T}}|11}(\theta, n) = F_2^{\mathcal{T}|11}(\theta, n), \tag{3.21}$$

$$F_2^{\tilde{\mathcal{T}}|1j}(\theta, n) = F_2^{\mathcal{T}|11}(\theta + 2\pi i(j-1), n). \tag{3.22}$$

3.2 The Ising Model

The Ising model is, together with the free Boson theory, the simplest integrable model we can possibly consider. It describes a real free fermion and therefore the scattering matrix is simply

$$S(\theta) = -1. \tag{3.23}$$

Form factors of local fields of the Ising model were first computed in [1, 23] and later on in [24] for so-called descendant fields. A solution of (3.9) for $j = k = 1$ is given by

$$F_{\min}^{\mathcal{T}|11}(\theta) = -i \sinh\left(\frac{\theta}{2n}\right). \tag{3.24}$$

This is in fact the standard minimal form factor already employed in [1, 23], with $\theta \to \theta/n$.

3.3 The Sinh-Gordon Model

The sinh-Gordon model is a quantum integrable model possessing a single particle spectrum and no bound states. The corresponding S-matrix [25–27] is given by

$$S(\theta) = \frac{\tanh\frac{1}{2}(\theta - i\frac{\pi B}{2})}{\tanh\frac{1}{2}(\theta + i\frac{\pi B}{2})}. \tag{3.25}$$

The parameter $B \in [0, 2]$ is the effective coupling constant which is related to the coupling constant β in the sinh-Gordon Lagrangian [28, 29] as

$$B(\beta) = \frac{2\beta^2}{8\pi + \beta^2}, \tag{3.26}$$

under CFT normalization [30]. The S-matrix is obviously invariant under the transformation $B \to 2 - B$, a symmetry which is also referred to as week-strong coupling duality, as it corresponds to $B(\beta) \to B(8\pi\beta^{-1})$ in (3.26). The point $B = 1$ is known as the self-dual point. Form factors of the sinh-Gordon model were first computed in [31]. The program was

thereafter extended to other operators in [32] and more recently in [33]. The S-matrix above admits an integral representation which is given by (3.14), with

$$g(t) = \frac{8\sinh(\frac{tB}{4})\sinh(\frac{t}{2}(1-\frac{B}{2})\sinh(\frac{t}{2})}{\sinh t}. \tag{3.27}$$

Therefore, the minimal form factor is given by

$$F_{\min}^{\mathcal{T}|11}(\theta) = \exp\left[-2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{tB}{4}\sinh\frac{t(2-B)}{4}}{\sinh(nt)\cosh\frac{t}{2}}\cosh t\left(n+\frac{i\theta}{\pi}\right)\right], \tag{3.28}$$

where we have chosen the normalization $\mathcal{N} = F_{\min}^{\mathcal{T}|11}(i\pi n)$. Employing the identity

$$\int_0^\infty \frac{dt}{t}\frac{\sinh(\alpha t)\sinh(\beta t)e^{-\gamma t}}{\sinh(ut)} = \frac{1}{2}\log\left[\frac{\Gamma(\frac{\alpha+\beta+\gamma+u}{2u})\Gamma(\frac{-\alpha-\beta+\gamma+u}{2u})}{\Gamma(\frac{-\alpha+\beta+\gamma+u}{2u})\Gamma(\frac{\alpha-\beta+\gamma+u}{2u})}\right], \tag{3.29}$$

where $\Gamma(x)$ is Euler's gamma function, we obtain the alternative representation

$$\begin{aligned}\log(F_{\min}^{\mathcal{T}|11}(\theta)) = \sum_{k=0}^{\infty}(-1)^k \log\Bigg[&\frac{\Gamma(\frac{2n-2w+B+2k}{4n})\Gamma(\frac{2n+2w+B+2k}{4n})}{\Gamma(\frac{n-w+k}{2n})\Gamma(\frac{n+w+k}{2n})}\\ &\times\frac{\Gamma(\frac{2n-2w+2-B+2k}{4n})\Gamma(\frac{2n+2w+2-B+2k}{4n})}{\Gamma(\frac{n-w+k+1}{2n})\Gamma(\frac{n+w+k+1}{2n})}\Bigg]\end{aligned} \tag{3.30}$$

with $w = n + i\theta/\pi$, or equivalently

$$\begin{aligned}F_{\min}^{\mathcal{T}|11}(\theta) = \prod_{k=0}^{\infty}\Bigg[&\frac{\Gamma(\frac{2n-2w+B+4k}{4n})\Gamma(\frac{2n+2w+B+4k}{4n})\Gamma(\frac{2n-2w+2-B+4k}{4n})\Gamma(\frac{2n+2w+2-B+4k}{4n})}{\Gamma(\frac{n-w+2k}{2n})\Gamma(\frac{n+w+2k}{2n})}\\ &\times\frac{\Gamma(\frac{n-w+2k+2}{2n})\Gamma(\frac{n+w+2k+2}{2n})}{\Gamma(\frac{2n-2w+B+4k+2}{4n})\Gamma(\frac{2n+2w+B+4k+2}{4n})\Gamma(\frac{2n-2w+4-B+4k}{4n})\Gamma(\frac{2n+2w+4-B+4k}{4n})}\Bigg].\end{aligned} \tag{3.31}$$

As a consistency check, it is quite easy to show that for $n = 1$ the minimal form factor above is the standard minimal form factor associated to local fields in the sinh-Gordon model computed in [31]. In Appendix 1 we will show how the same expression can be derived form the angular quantization scheme proposed in [34] and later carried out for the exponential fields of various models (including the sinh-Gordon model) in [35].

4 Identifying the Ultraviolet Conformal Dimension of $\mathcal{T}$

In this section we verify that the form factors constructed above agree with the properties of the operator $\mathcal{T}$ at conformal level, that is, in the ultraviolet limit. As is well-known, the form factor program provides a way of carrying out this verification by allowing us to compute (at least in an approximate way) the correlation functions of various fields of an integrable quantum field theory. In the ultraviolet limit, it is possible to relate a particular correlation

function to the holomorphic conformal dimension of a primary field by means of the so-called Δ-sum rule:

$$\Delta^{\mathcal{T}} = \Delta^{\tilde{\mathcal{T}}} = -\frac{1}{2\langle \mathcal{T}\rangle}\int_0^\infty r\langle \Theta(r)\tilde{\mathcal{T}}(0)\rangle dr \tag{4.1}$$

(where the integration is on a space-like ray), originally proposed by G. Delfino, P. Simonetti and J.L. Cardy in [36], where Θ is the local operator corresponding to the trace of the stress-energy tensor. The first equality, expected from CFT, holds from the Δ-sum rule thanks to the fact that Θ commute with $\mathcal{T}$ and that $\Theta^\dagger = \Theta$. The holomorphic conformal dimension is related to the scaling dimension by $d_n = 2\Delta^{\mathcal{T}}$, where d_n is expected to be (2.8).

By introducing a sum over all quantum states and carrying out the r-integration, the expression above can be rewritten as

$$\begin{aligned}\Delta^{\mathcal{T}} = -\frac{1}{2\langle \mathcal{T}\rangle}\sum_{k=1}^\infty \sum_{\mu_1\cdots\mu_k}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{d\theta_1\cdots d\theta_k}{k!(2\pi)^k(\sum_{i=1}^k m_{\mu_i}\cosh\theta_i)^2} \\ \times F_k^{\Theta|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k)(F_k^{\mathcal{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k))^*,\end{aligned} \tag{4.2}$$

where the sum in μ_i with $i = 1,\ldots,k$ is a sum over particle types in the theory under consideration. The sum starts at $k = 1$ since we are considering "connected" correlation functions, that is, the $k = 0$ contribution has been subtracted. The sum above, can only be carried out in particularly simple cases. For most models, one must be content with evaluating just the first few contributions to the sum. Fortunately, the many studies carried out in the last years provide strong evidence that the sum above is convergent and that in fact, the first few terms provide the main contribution to the final result. Indeed, the convergence is often so good that considering only the contribution with $k = 2$ already provides very precise results (see e.g. [31]). Expecting a similar behaviour also in our case, we will approximate the sum above by the two-particle contribution, that is

$$\Delta^{\mathcal{T}} \approx -\frac{n}{2\langle \mathcal{T}\rangle}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{d\theta_1 d\theta_2 F_2^{\Theta|11}(\theta_{12})F_2^{\mathcal{T}|11}(\theta_{12},n)^*}{2(2\pi)^2m^2(\cosh\theta_1+\cosh\theta_2)^2}. \tag{4.3}$$

The factor of n is a consequence of summing over all particle types and using (3.10). In addition, the only non-vanishing contribution comes from form factors involving only one particle type, since we are considering n non-interacting copies of the model. This implies that

$$F_2^{\Theta|ij}(\theta) = 0, \quad \forall i \neq j. \tag{4.4}$$

Changing variables to $\theta = \theta_1 - \theta_2$ and $\theta' = \theta_1 + \theta_2$ we obtain,

$$\begin{aligned}\Delta^{\mathcal{T}} &\approx -\frac{n}{2\langle \mathcal{T}\rangle}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{d\theta d\theta' F_2^{\Theta|11}(\theta)F_2^{\mathcal{T}|11}(\theta,n)^*}{2(2\pi)^2m^2(2\cosh(\theta/2)\cosh(\theta'/2))^2} \\ &= -\frac{n}{32\pi^2m^2\langle \mathcal{T}\rangle}\int_{-\infty}^\infty d\theta\,\frac{F_2^{\Theta|11}(\theta)F_2^{\mathcal{T}|11}(\theta,n)^*}{\cosh^2(\theta/2)}.\end{aligned} \tag{4.5}$$

Let us now evaluate this integral both for the Ising and sinh-Gordon models.

4.1 The Ising Model

For the Ising model the only non-vanishing form factor of the trace of the stress-energy tensor is the 2-particle form factor. Hence, the two-particle approximation (4.3) becomes exact. The two-particle form factors are given by

$$F_2^{\mathcal{T}|11}(\theta) = \frac{-i\langle\mathcal{T}\rangle\cos(\frac{\pi}{2n})}{n\sinh(\frac{i\pi+\theta}{2n})\sinh(\frac{i\pi-\theta}{2n})}\sinh\left(\frac{\theta}{2n}\right), \qquad F_2^{\Theta|11}(\theta) = -2\pi i m^2\sinh\left(\frac{\theta}{2}\right), \quad (4.6)$$

and therefore

$$\Delta^{\mathcal{T}} = -\frac{1}{16\pi}\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi}{2n})\sinh(\frac{\theta}{2n})\sinh(\frac{\theta}{2})}{\sinh(\frac{i\pi+\theta}{2n})\sinh(\frac{i\pi-\theta}{2n})\cosh^2(\frac{\theta}{2})}\,d\theta. \quad (4.7)$$

It is easy to check numerically that the above integral exactly reproduces the expected value (2.8) for $c = 1/2$,

$$2\Delta^{\mathcal{T}} = \frac{1}{24}\left(n - \frac{1}{n}\right) = d_n, \quad (4.8)$$

for any value of n. The integral can also be computed analytically, at least for n even. In this case, shifting t by $2\pi n i$ the integral above changes by a sign and therefore it is possible to show

$$2\Delta^{\mathcal{T}} = 2\pi i\sum_{j=1}^{n} r_j, \quad (4.9)$$

where r_j are the residues of the poles of the integrand at $t = i\pi(2j-1)$, with $j = 1,\ldots,n$. Of those, the poles at $j = 1, n$ are triple whereas all the others are double poles. A tedious but straightforward computation yields

$$\begin{aligned}
2\pi i r_1 &= \frac{-1+n^2+6\cot(\frac{\pi}{n})/\sin(\frac{\pi}{n})}{48n},\\
2\pi i r_j &= \frac{(-1)^{j+1}(\cot(\frac{(j-1)\pi}{n})/\sin(\frac{(j-1)\pi}{n}) + \cot(\frac{j\pi}{n})/\sin(\frac{j\pi}{n}))}{8n}, \quad 1 < j < n,\\
2\pi i r_n &= \frac{(-1)^{n+1}(4-n^2-3\cot^2(\frac{\pi}{2n})+6\cot^2(\frac{\pi}{n}))}{48n}.
\end{aligned} \quad (4.10)$$

Finally, we need to add up all these residues. The sum over the r_j residues becomes in fact very simple, since it is a telescopic series. We obtain,

$$2\pi i\sum_{j=2}^{n-1} r_j = -\frac{\cot(\frac{\pi}{n})}{4n\sin(\frac{\pi}{n})}, \quad (4.11)$$

which gives (4.8).

4.2 The Sinh-Gordon Model

In this case, the relevant 2-particle form factors are given by

$$F_2^{\mathcal{T}|11}(\theta) = \frac{\langle \mathcal{T}\rangle \sinh(\frac{\pi}{n})}{2n\sinh(\frac{i\pi+\theta}{2n})\sinh(\frac{i\pi-\theta}{2n})}\frac{F_{\min}^{\mathcal{T}|11}(\theta,n)}{F_{\min}^{\mathcal{T}|11}(i\pi,n)}, \qquad F_2^{\Theta|11}(\theta) = 2\pi m^2 \frac{F_{\min}^{\mathcal{T}|11}(\theta,1)}{F_{\min}^{\mathcal{T}|11}(i\pi,1)}. \tag{4.12}$$

The form factors of Θ were computed in [31]. Since Θ is a local operator, its minimal form factor is given by (3.31) with $n=1$. Thus,

$$\Delta^{\mathcal{T}} \approx -\frac{1}{32\pi}\int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{n})}{\sinh(\frac{i\pi+\theta}{2n})\sinh(\frac{i\pi-\theta}{2n})\cosh^2(\frac{\theta}{2})}\frac{F_{\min}^{\mathcal{T}|11}(\theta,n)^*}{F_{\min}^{\mathcal{T}|11}(i\pi,n)^*}\frac{F_{\min}^{\mathcal{T}|11}(\theta,1)}{F_{\min}^{\mathcal{T}|11}(i\pi,1)}d\theta. \tag{4.13}$$

The tables below show the result of carrying out this integral numerically for various values of n and B. Next to each value of n in brackets we show for reference the expected value of $\Delta^{\mathcal{T}}$, as predicted by the CFT formula (2.8) (with, again, $\Delta^{\mathcal{T}} = d_n/2$).

	$n=2$ (0.0625)	$n=3$ (0.1111)	$n=4$ (0.1563)	$n=5$ (0.2)
$B=0.02$	0.0620	0.1114	0.1567	0.2007
$B=0.2$	0.0636	0.1135	0.1599	0.2048
$B=0.4$	0.0636	0.1148	0.1620	0.2074
$B=0.6$	0.0643	0.1155	0.1631	0.2088
$B=0.8$	0.0644	0.1158	0.1636	0.2096
$B=1$	0.0644	0.1159	0.1637	0.2098

	$n=6$ (0.2431)	$n=7$ (0.2857)	$n=8$ (0.3281)	$n=9$ (0.3704)	$n=10$ (0.4125)
$B=0.02$	0.2436	0.2864	0.3289	0.3712	0.4135
$B=0.2$	0.2488	0.2925	0.3360	0.3793	0.4225
$B=0.4$	0.2522	0.2966	0.3407	0.3846	0.4284
$B=0.6$	0.2540	0.2988	0.3433	0.3876	0.4317
$B=0.8$	0.2550	0.2999	0.3446	0.3890	0.4334
$B=1$	0.2552	0.3002	0.3449	0.3895	0.4339

The figures obtained are extremely close to their expected value for all choices of B and n. In most cases they are slightly above the expected value. This is not surprising since the 4- and higher particle contributions are not necessarily positive.

5 Two-Point Functions and the Entanglement Entropy

As explained before, the entanglement entropy is given by the derivative with respect to n of the two-point function $\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle$ evaluated at $n=1$. The behaviour of the entropy at short separations $r \ll m^{-1}$ is described by the conformal limit of the model and is already

well known [18, 19]. At large separations $r \gg m^{-1}$ (in the infrared limit) it is also known to saturate. Here we would like to evaluate the first correction to the entropy in the infrared limit. In this limit, the two-particle contribution provides the first sub-leading exponential term in the correlation function $\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle$. Hence, the two-particle approximation should provide both the saturation value of the entropy, coming from the disconnected part of the correlation function, and the exact first exponential correction at large rm, coming from the two-particle contributions.

The two-point function in the two-particle approximation is given by

$$\begin{aligned}\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle &\approx \langle \mathcal{T}\rangle^2 + \sum_{i,j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{2!(2\pi)^2}|F_2^{\mathcal{T}|ij}(\theta_{12},n)|^2\, e^{-rm(\cosh\theta_1+\cosh\theta_2)}\\ &= \langle \mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty}d\theta f(\theta,n)K_0(2rm\cosh(\theta/2))\right),\end{aligned} \tag{5.1}$$

where we changed variables as in Sect. 4, $K_0(z)$ is the Bessel function resulting from carrying out one of the integrals, and we defined

$$\begin{aligned}\langle \mathcal{T}\rangle^2 f(\theta,n) &= \sum_{j=1}^{n}|F_2^{\mathcal{T}|1j}(\theta,n)|^2\\ &= |F_2^{\mathcal{T}|11}(\theta,n)|^2 + \sum_{j=1}^{n-1}|F_2^{\mathcal{T}|11}(-\theta+2\pi ij,n)|^2\end{aligned} \tag{5.2}$$

(with $f(\theta,1)=0$). Notice that the function above is only defined for integer values of n.

In order to obtain the entropy we should now analytically continue the two-point function (5.1), as function of rm and n, from $n=1,2,3,\ldots$ to $n\in[1,\infty)$, compute the derivative with respect to n and evaluate the result at $n=1$. The analytic continuation is of course not unique. We will choose the one which is such that for $\mathrm{Re}(n)>0$, the two-point function divided by $\langle \mathcal{T}\rangle^2$ is $O(e^{qn})$ as $n\to\infty$ for some $q<\pi$. It is then unique by Carlson's theorem [37]. This choice is motivated by the fact that the trace (2.11) has this behavior with $q<0$ for any finite system, since the eigenvalues of ρ_A are real an positive, and are normalised to sum to 1. Of course, this is merely a motivation. For infinite systems, eigenvalues should have dense components and we could have algebraic behaviors; the scaling limit and the limit $n\to\infty$ may not commute, so that the coefficient of the small-distance power-law ϵ^{2d_n} may be divergent; and the limit $rm\to\infty$ and $n\to\infty$ of this coefficient may also not commute, so that the behavior in n of the large-rm expansion coefficients may also be divergent. We expect that this gives at most algebraic divergencies in n, but a better understanding would be desirable.

There are three main observations necessary to understand the analytic continuation of (5.1) and the evaluation of the derivative at $n=1$:

- *Structure of the analytic continuation* There is no natural (as described above) analytic continuation of $f(0,n)$ from $n=1,2,3,\ldots$ to $n\in[1,\infty]$. Instead, there is such an analytic continuation from $n=2,3,4,\ldots$ to $n\in[1,\infty]$, which we will denote by $\tilde{f}(n)$, but it has the property that

$$\tilde{f}(1)\neq f(0,1)=0. \tag{5.3}$$

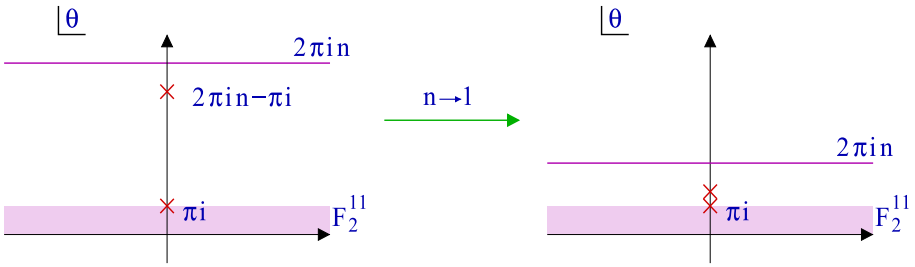

Fig. 6 (Color online) The collision of kinematic singularities when $n \to 1$

On the other hand, the function $f(\theta, n)$ does not have any such feature for $\theta \neq 0$: its analytic continuation $\tilde{f}(\theta, n)$ to $n \in [1, \infty]$ agrees with $f(\theta, n)$ for all $n = 1, 2, 3, \ldots$. In particular, $\tilde{f}(\theta, 1) = 0$. Hence, as $n \to 1$, the function $\tilde{f}(\theta, n)$ does not converge uniformly on $\theta \in (-\infty, \infty)$.

- *Kinematic singularities* The non-uniform convergence of $\tilde{f}(\theta, n)$ can be seen to be a consequence of the collision of the kinematic singularities of form factors $F_2^{\mathcal{T}|11}(\theta, n)$ (at $\theta = i\pi$ and $\theta = i\pi(2n-1)$) that occur when $n \to 1$ (see Fig. 6). The idea is made clear from considering Poisson's resummation formula. Consider a sum of the type $\sum_{j=1}^{n-1} s(\theta, j)$, as in the r.h.s. of (5.2), re-written using Poisson's resummation formula, which holds whenever $s(\theta, n) = s(\theta, 0)$ (the summand $s(\theta, j)$ in (5.2) indeed satisfies this by unitarity):

$$\sum_{j=1}^{n-1} s(\theta, j) = \sum_{k=-\infty}^{\infty} (s_{nk} - s_k), \quad s_k = \int_0^n dj\, e^{-\frac{2\pi i j k}{n}} s(\theta, j). \tag{5.4}$$

For any $\theta \neq 0$, no singularity of $s(\theta, j)$ occur on the integration path[4] defining s_k, so that when $n \to 1$, the result should vanish. In fact, since $s(\theta, j)$ itself vanishes like $(n-1)^2$ for $\theta \neq 0$, the result vanishes like $(n-1)^3$. When $\theta \to 0$, the poles of $s(\theta, j)$, which are at $j = \frac{1}{2} \pm \frac{\theta}{2\pi i}$ and $j = n - \frac{1}{2} \pm \frac{\theta}{2\pi i}$, pinch the integration path, but the quantities s_k are still finite because the divergent contributions at $\mathrm{Re}(j) = \frac{1}{2}$ and $\mathrm{Re}(j) = n - \frac{1}{2}$ cancel out—we are left with principal-value integrals with two double-poles (and the resulting conditionally convergent sum over k has a unique finite value defined by the limit $\theta \to 0$). However, since these two double poles collide at $n = 1$ and fuse into a higher order pole, there is no guarantee that the result vanishes as $n \to 1$.

- *Delta-function* From the arguments above, the full contributions of the derivative with respect to n of the two-particle contribution to the two-point function, in the limit $n \to 1$, is obtained from the region $\theta \sim 0$, and is due to the kinematic singularities. In fact, notice that as $n \to 1$, the derivative with respect to n of $\tilde{f}(\theta, n)$ "around" $\theta = 0$ should diverge: indeed, the values of $\tilde{f}(\theta, n)$ at $\theta = 0$ and at its neighboring points should be about the same for all $n > 1$, by continuity, but they should reach a finite separation at $n = 1$; hence variations must be very strong near to $n = 1$. This leads to the expectation that $\frac{\partial}{\partial n} \tilde{f}(\theta, n)$ at $n = 1$ is proportional to $\delta(\theta)$. The contribution of the kinematic singularities to the sum

[4]The sum over k can be made absolutely convergent by a slight imaginary shift of the j-integration path defining s_k. Since $s(\theta, j)$ is in fact not periodic in j, there are contributions along $\mathrm{Re}(j) = 0, n$ additionally to the shifted path, with coefficients that vanish linearly as $\mathrm{Im}(j) \to 0$. This linear vanishing guarantees a vanishing like $1/k^2$ of s_k at large k, making the sum convergent.

in the function $\tilde{f}(\theta, n)$ (5.2) is obtained from the singular behavior in j of the summand $s(\theta, j) = F_2^{\mathcal{T}|11}(-\theta + 2\pi i j, n)(F_2^{\mathcal{T}|11})^*(-\theta - 2\pi i j, n)$:

$$s(\theta, j) \sim \frac{i F_2^{\mathcal{T}|11}(-2\theta + 2\pi i n - i\pi)}{-\theta - 2\pi i j + 2\pi i n - i\pi} - \frac{i F_2^{\mathcal{T}|11}(-2\theta + i\pi)}{-\theta - 2\pi i j + i\pi} + \text{c.c.}$$

(where c.c. means complex conjugate, for real θ). It is a simple matter to perform on this expression the sum $\sum_{j=1}^{n-1}$, giving

$$\begin{aligned}\sum_{j=1}^{n-1} s(\theta, j) \sim{}& \frac{1}{2\pi}\left(\psi\left(-\frac{1}{2} + n + \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right)\right) F_2^{\mathcal{T}|11}(-2\theta + 2\pi i n - i\pi) \\ &+ \frac{1}{2\pi}\left(\psi\left(-\frac{1}{2} + n - \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)\right) F_2^{\mathcal{T}|11}(-2\theta + i\pi) \\ &+ \text{c.c.},\end{aligned}$$

where $\psi(z) = d \log \Gamma(z)/dz$ is the derivative of the logarithm of Euler's Gamma function. This has no poles at $\theta = 0$, as the kinematic poles of the form factors involved cancel out. The poles that are nearest to $\mathrm{Re}(\theta) = 0$ as $n \to 1$ are at $\theta = \pm i\pi(n-1)$, coming from the form factors involved. The residues to first order in $n - 1$ give:

$$\tilde{f}(\theta, n) \sim \tilde{f}(1)\left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))}\right) \quad (n \to 1) \tag{5.5}$$

with

$$\tilde{f}(1) = \frac{1}{2}. \tag{5.6}$$

This has simple poles at $\theta = \pm i\pi(n-1)$ with residues that vanish at $n = 1$, gives $\tilde{f}(1)$ at $\theta = 0$ and vanishes like $(n-1)^2$ as $n \to 1$ for $\theta \neq 0$. The limit $n \to 1$, as a distribution on θ, is easily evaluated:

$$\left(\frac{\partial}{\partial n}\tilde{f}(\theta, n)\right)_{n=1} = \pi^2 \tilde{f}(1)\delta(\theta). \tag{5.7}$$

In Appendix 3, we give the full form of $\tilde{f}(\theta, n)$ and verify that this is correct. Note that for the free case our result is in agreement with the $n \to 1$ limit evaluated in [21].

Inserting this inside (5.1) gives, using (2.14) and (2.12), the entanglement entropy $S_A(rm)$ for A an interval of length r:

$$S_A(rm) = -\frac{c}{3}\log(\epsilon m) + U - \frac{1}{8}K_0(2rm) + O(e^{-4rm}), \tag{5.8}$$

where

$$U = -\frac{d}{dn}(m^{-2d_n}\langle\mathcal{T}\rangle^2)_{n=1}. \tag{5.9}$$

Therefore the sub-leading large rm terms in the entropy are given by the Bessel function $K_0(2rm)$, up to terms that are exponentially smaller (coming from the neglected 4-particle form factors). The term U involving the derivative of the vacuum expectation

value of $\mathcal{T}$ has a universal meaning since the normalisation of $\mathcal{T}$ has been fixed (see (3.8)). More precisely, with this normalisation, the entanglement entropy at short interval length is $S_A(rm) = -\frac{c}{3}\log(\epsilon/r) + O(rm)$. It is of course possible to define the quantity U in a way that is obviously universal, valid for any choice of short-distance cutoff:

$$U = \lim_{\xi\to\infty}\left(S_A(\xi) - S_A(\xi^{-1}) - \frac{c}{3}\log\xi\right). \tag{5.10}$$

We will now proceed to evaluate explicitly the functions $f(0,n)$ and $\tilde{f}(n)$ for the Ising and sinh-Gordon models, verifying some of the results above. It is worth noting that the function $f(0,n)$, which we study in more detail below, also has a meaning as the coefficient of the leading exponential correction to the partition function on the Riemann surface $\mathcal{M}_{n,0,r}$:

$$\langle\mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle = \langle\mathcal{T}\rangle^2\left(1 + \frac{nf(0,n)e^{-2rm}}{4\pi rm} + O\left(\frac{e^{-2rm}}{(rm)^2}\right)\right). \tag{5.11}$$

5.1 The Ising Model

For the Ising model, the function (5.2) at $\theta = 0$ is given by

$$f(0,n) = \frac{\cos^2(\frac{\pi}{2n})}{n^2}\sum_{j=2}^{n}\frac{\sin^2(\frac{(j-1)\pi}{n})}{\sin^2(\frac{(2j-1)\pi}{2n})\sin^2(\frac{(2j-3)\pi}{2n})} = \frac{1}{2}. \tag{5.12}$$

The result of the sum can be obtained analytically as in Appendix 3. In this case, the integral part of the formula (10.6) is zero, so that $\tilde{f}(n) = 1/2$ for all n. That is, the connected part of the correlation function at large rm behaves linearly with n, for all values of n. Hence, the entanglement entropy can be computed to

$$S_A(rm) = -\frac{1}{6}\log(\epsilon m) + U_{\text{Ising}} - \frac{1}{8}K_0(2rm) + O(e^{-4rm}). \tag{5.13}$$

The constant U_{Ising} can also be evaluated explicitly using the relation between the Ising model and the free Dirac fermionic model, as explained in Appendix 2. We find that the expectation value of the branch-point twist field is

$$\langle\mathcal{T}\rangle = \left(\frac{m}{2}\right)^{\frac{1}{24}(n-\frac{1}{n})}\exp\left[\int_0^\infty\frac{dt}{4t}\left(\frac{1}{\sinh t\sinh\frac{t}{n}} - \frac{n}{\sinh^2 t} - \frac{e^{-2t}}{6}\left(n-\frac{1}{n}\right)\right)\right], \tag{5.14}$$

which gives

$$U_{\text{Ising}} = \frac{1}{6}\log 2 - \int_0^\infty\frac{dt}{2t}\left(\frac{t\cosh t}{\sinh^3 t} - \frac{1}{\sinh^2 t} - \frac{e^{-2t}}{3}\right) = -0.131984\ldots. \tag{5.15}$$

Once the value of U_{Ising} has been fixed we can compare our expressions for the entropy in the deep-infrared and deep-ultraviolet regimes to existing results in the literature for the quantum Ising chain. The comparison goes as follows: from [19, 38] it is possible to obtain the expression of the entropy at large separations $r \gg m^{-1}$ in terms of the lattice spacing a

$$S_A = -\frac{1}{6}\log(am) + \frac{1}{2}\log 2. \tag{5.16}$$

By comparing (5.16) to our formula

$$S_A = -\frac{1}{6}\log(\epsilon m) + U_{\text{Ising}}, \tag{5.17}$$

we obtain the precise relationship between the short-distance cutoff ϵ and the lattice spacing a. That is

$$\epsilon = \frac{a}{8} e^{6U_{\text{Ising}}} = (0.0566227\ldots)a. \tag{5.18}$$

As mentioned above, formula (5.16) follows from the results in [19, 38]. However this is not entirely trivial and some clarifications are due here. First, the formulae given in these publications are expressed in terms of a parameter k rather than a. The parameter k is related to the value of the transverse magnetic field h of the quantum Ising chain as follows:

$$k = \begin{cases} h, & \text{for } h < 1, \\ h^{-1}, & \text{for } h > 1, \end{cases} \tag{5.19}$$

where $h = 1$ corresponds to the critical point. The entropy in both regions of values of k was computed in [38] whereas in [19] only the $h < 1$ regime was considered. However, it is easy to show that both regimes give the same infrared value of the entropy when h approaches 1. Second, it is a standard result that $\pm am = 1 - h$, where the positive sign corresponds to $h < 1$ and the negative sign corresponds to $h > 1$. Therefore, the fact that the infrared value of the entropy is the same both for $h > 1$ and $h < 1$ is in agreement with what we expect from QFT, as going from one regime to the other amounts formally to a change in the sign of the mass m which has no effect on the value of the energy density. That is, the two regions should be described by the same QFT in the scaling limit. The relations between k and h and between am and h given above allow us to relate the parameter k (in terms of which the entropy is expressed) to the lattice spacing a. Once this is done we only need to expand the expressions in [19, 38] around the value $k = 1$. Since these formulae are given in terms of the complete elliptic integral of the first kind $K(k)$, we need to employ the standard expansion

$$K(k) \sim -1/2\log(1-k) + 3/2\log 2, \quad \text{for } k \to 1, \tag{5.20}$$

to obtain (5.16).

Plugging (5.18) into the short-distance expression of the entropy we obtain:

$$S_A = -\frac{1}{6}\log(\epsilon/r) + O(rm) = -\frac{1}{6}\log(a/r) + 0.478558\ldots + O(rm). \tag{5.21}$$

We can compare this result to the numerical values obtained in [11] for the Ising spin chain. However, we must notice first that in there the entropy was defined as

$$S_L = -\operatorname{Tr}_{\mathcal{H}_A}(\rho_A \log_2(\rho_A)), \tag{5.22}$$

whereas in this paper we have used the definition (1.2). This means that in order to compare (5.21) to the results obtained in [11] we must divide our formula by a $\log 2$ factor. This gives

$$S_L = \frac{1}{6}\log_2(L) + 0.690413\ldots, \tag{5.23}$$

where we now introduced the parameter $L = r/a$ which is the number of sites in the interval A. A plot of this function is presented in Fig. 7 which is to be compared to the blue curve

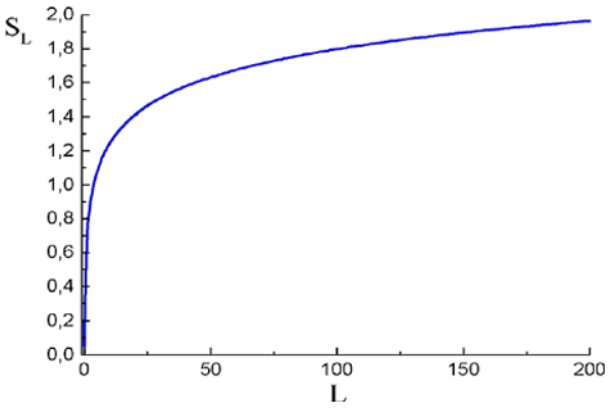

Fig. 7 (Color online) A plot of the function $S_L = \frac{1}{6}\log_2(L) + 0.690413$ for $L \leq 200$. The graph is in very good agreement with the numerical values plotted in Fig. 9 of [11]

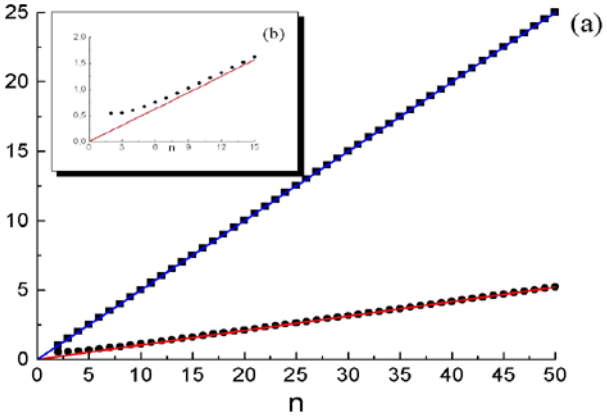

Fig. 8 (Color online) **a** shows 4 functions: the points are the function $nf(0,n)$ for integer values of n in the interval $[2, 50]$ both for the Ising (*black squares*) and sinh-Gordon (*black circles*) models, evaluated numerically. The *solid blue line* gives the corresponding analytic continuation $n\tilde{f}(n)$ for real values of n in the interval $[0, 50]$ for the Ising model, that is the function $n/2$. Finally the *solid red line* gives the function $n\tilde{f}(\infty)$ for the sinh-Gordon model, that is a straight line passing through the origin which describes the asymptotic behaviour of the function $n\tilde{f}(n)$ for n large. In the sinh-Gordon case, all functions have been computed for $B = 0.5$. **b** is a magnification of the *lower left corner* of the sinh-Gordon part of (**a**)

in Fig. 9 of [11]. Very good agreement is found, which supports the twist-field realization proposed in Appendix 2.

It is worth noticing that in the free case there are alternative ways of computing the entanglement entropy [39–41].

5.2 The Sinh-Gordon Model

In this section we will verify that $\tilde{f}(1) = 1/2$ also for the sinh-Gordon model. Here, the integral part of (10.6) is not vanishing, hence in contrast to the Ising model result (5.12), $\tilde{f}(n)$ is in general not constant with n. This can be easily seen from Fig. 8 where the functions $nf(0,n)$ and $n\tilde{f}(\infty)$ are presented, both for the Ising and sinh-Gordon models.

The non-linearity of the function $n\tilde{f}(n)$ in the sinh-Gordon model can be seen more clearly by magnifying the lower left corner of Fig. 8a, which we have done in Fig. 8b.

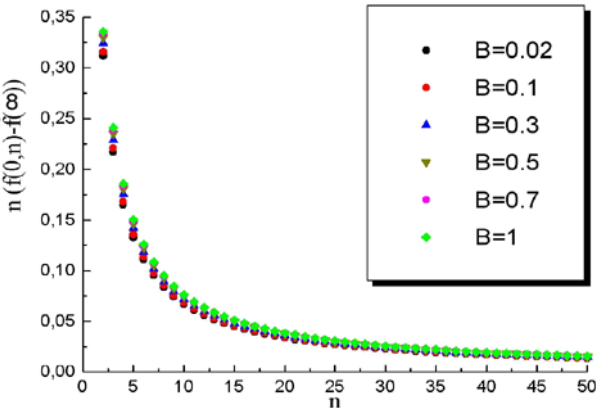

Fig. 9 (Color online) The function $n(f(0,n)-\tilde{f}(\infty))$ for integer n in the interval $[2,50]$, for various values of B

Figure 8b also shows how $\tilde{f}(1)$ appears to tend to the value $1/2$ and how the function $n\tilde{f}(n)$ deviates from the straight line for small values of n. Both for the Ising and sinh-Gordon model the figures show clearly that the function $nf(0,n)$ has a jump at $n=1$, since $f(0,1)=0$. We will now attempt to provide a more rigorous description of these behaviours.

In principle we would only need to perform the integral in (10.6) to obtain $\tilde{f}(n)$ for any values of n. However this is highly non-trivial due to the complexity of the minimal form factor, as a function of n. We choose therefore to proceed in a different and more instructive way: we will instead find the natural analytic continuation of the function $nf(0,n)$ numerically, as a large n expansion in powers of $1/n$.

First of all, it is an interesting exercise to try to determine analytically the precise slope of the line $n\tilde{f}(\infty)$ in the sinh-Gordon model. This is a relatively tedious but straightforward computation which we present in Appendix 4. The main result is the value of $\tilde{f}(\infty)$ as a function of B, which is given by

$$\begin{aligned}\tilde{f}(\infty) = {} & \frac{8192}{\pi^2(4-B)^2(2+B)^2}\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\kappa-1)\Gamma(\sigma-1)}\right]^4\left[\frac{1}{9}\,{}_7F_6\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1,\alpha,\alpha,\beta,\beta;1\\ \frac{5}{2},\frac{5}{2},\kappa,\kappa,\sigma,\sigma\end{matrix}\right]\right.\\ & +\frac{1}{75}\left(\frac{\alpha\beta}{\kappa\sigma}\right)^2\left({}_7F_6\left[\begin{matrix}\frac{3}{2},\frac{3}{2},2,\alpha+1,\alpha+1,\beta+1,\beta+1;1\\ \frac{7}{2},\frac{7}{2},\kappa+1,\kappa+1,\sigma+1,\sigma+1\end{matrix}\right]\right.\\ & \left.\left.+\frac{6(8+B)^2(B-10)^2}{49(10+B)^2(B-12)^2}\,{}_7F_6\left[\begin{matrix}\frac{5}{2},\frac{5}{2},3,\alpha+2,\alpha+2,\beta+2,\beta+2;1\\ \frac{9}{2},\frac{9}{2},\kappa+2,\kappa+2,\sigma+2,\sigma+2\end{matrix}\right]\right)\right],\end{aligned} \tag{5.24}$$

in terms of the generalized hypergeometric functions defined in (11.7) and the variables

$$\alpha=\frac{3}{2}-\frac{B}{4},\qquad \beta=1+\frac{B}{4},\qquad \kappa=2-\frac{B}{4},\qquad \sigma=\frac{3}{2}+\frac{B}{4}. \tag{5.25}$$

As mentioned before, the data in Fig. 8b also exhibit a clear deviation from the linear behaviour for small values of n, which is the region we are most interested in. Figure 9 shows the values of the function $nf(0,n)$ for integer n in the interval $[2,50]$ with the linear part, that is $n\tilde{f}(\infty)$ subtracted, for various values of B. By subtracting the linear part, the behav-

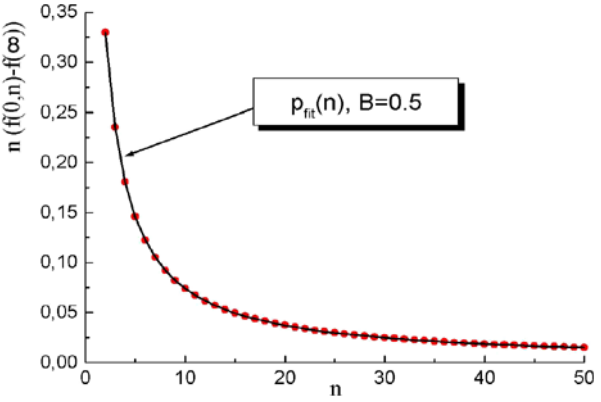

Fig. 10 (Color online) For $B = 0.5$, $p_{\rm fit}(n)$ (5.26), and $n(f(0,n) - \tilde{f}(\infty))$ at integer values of n, in the interval [2, 50]

iour of the function for large n becomes more clear and appears to be dominated by a term proportional to $1/n$. This dependence can be made more precise by numerically fitting the various functions plotted above to a function of the generic form

$$p_{\rm fit}(n) = \frac{a_0}{n} + \frac{a_1}{n^3} + \frac{a_2}{n^5} + \frac{a_3}{n^7}. \tag{5.26}$$

The fact that only odd powers of n appear is a numerical observation. The table below contains the values of the constants a_0, a_1, a_2 and a_3 obtained numerically for various values of B, as well as the exact value of $\tilde{f}(\infty)$ from (5.24):

B	$\tilde{f}(\infty)$	a_0	a_1	a_2	a_3
0.02	0.0952	0.67(1)	−0.17(2)	−0.06(1)	−0.02(3)
0.1	0.0972	0.68(5)	−0.20(5)	−0.04(8)	−0.02(3)
0.2	0.0994	0.70(3)	−0.24(2)	−0.02(9)	−0.02(7)
0.3	0.1013	0.71(8)	−0.27(7)	−0.00(9)	−0.03(1)
0.4	0.1030	0.73(1)	−0.30(8)	0.01(0)	−0.03(8)
0.5	0.1044	0.74(2)	−0.33(5)	0.02(9)	−0.04(4)
0.6	0.1055	0.75(1)	−0.35(8)	0.04(5)	−0.05(1)
0.7	0.1064	0.75(8)	−0.37(6)	0.05(9)	−0.05(6)
0.8	0.1070	0.76(3)	−0.38(9)	0.06(9)	−0.06(1)
0.9	0.1074	0.76(6)	−0.39(7)	0.07(5)	−0.06(4)
1	0.1075	0.76(7)	−0.39(9)	0.07(7)	−0.06(5)

As an example, Fig. 10 shows the fit (5.26) for $B = 0.5$ together with the corresponding values of $n(f(0,n) - \tilde{f}(\infty))$ for integer values of n in the interval [2, 50]. The fit (5.26) is in fact extremely good for all values of B. More precisely, we have checked that the values of $n(f(0,n) - \tilde{f}(\infty)) - p_{\rm fit}(n)$ for integer n are always smaller than 10^{-6} for all values of B included in the previous table. We are now in the position to compute $\tilde{f}(1)$ numerically. Its value for a certain B corresponds simply to adding up all numbers in the corresponding row of the table above. The outcome of this computation is:

B	0.02	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\tilde{f}(1)$	0.5(1)	0.5(1)	0.5(0)	0.5(0)	0.5(0)	0.5(0)	0.4(9)	0.4(9)	0.4(9)	0.4(9)	0.4(9)

As expected, all values obtained agree with the predicted value of $1/2$ within our numerical precision. Hence, the entanglement entropy can be computed to

$$S_A(rm) = -\frac{1}{3}\log(\epsilon m) + U_{\text{sinh-Gordon}} - \frac{1}{8}K_0(2rm) + O(e^{-4rm}). \tag{5.27}$$

We do not know how to evaluate the constant U yet, but the methods developed by S. Lukyanov [42] may be helpful.

6 Generalization to Theories with Several Particles and Bound States

So far we have been dealing with theories with a single particle spectrum and no bound states. In this section we wish to investigate how our results for the entropy can be generalized to situations in which the spectrum of the initial integrable QFT consists of $\ell > 1$ particles and bound states may be present. As before we will consider n copies of the theory and therefore it will be natural to label particles by two indices

$$(\alpha, i) \quad \text{with } \alpha = 1, \dots, \ell \text{ and } i = 1, \dots, n, \tag{6.1}$$

with the identification $(\alpha, i) \equiv (\alpha, i+n)$ for all $i = 1, \dots, n$ and $\alpha = 1, \dots, \ell$. Denoting by $\Psi_{(\alpha,i)}$ some fundamental field of the theory related to particle (α, i), the exchange relations with the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ can be written similarly as before:

$$\Psi_{(\alpha,i)}(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_{(\alpha,i+1)}(y), \quad x^1 > y^1, \tag{6.2}$$

$$\Psi_{(\alpha,i)}(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_{(\alpha,i)}(y), \quad x^1 < y^1, \tag{6.3}$$

$$\Psi_{(\alpha,i)}(y)\tilde{\mathcal{T}}(x) = \tilde{\mathcal{T}}(x)\Psi_{(\alpha,i-1)}(y), \quad x^1 > y^1, \tag{6.4}$$

$$\Psi_{(\alpha,i)}(y)\tilde{\mathcal{T}}(x) = \tilde{\mathcal{T}}(x)\Psi_{(\alpha,i)}(y), \quad x^1 < y^1. \tag{6.5}$$

Denoting by $S_{\alpha\beta}(\theta)$ with $\alpha, \beta = 1, \dots, \ell$ the two-particle S-matrix of the original theory, the S-matrix of the n-sheeted theory can be written as:

$$S_{(\alpha,i)(\beta,i)}(\theta) = S_{\alpha\beta}(\theta), \quad \forall \alpha, \beta, i, \tag{6.6}$$

$$S_{(\alpha,i)(\beta,j)}(\theta) = 1, \quad \forall \alpha, \beta, i, j \text{ with } i \neq j. \tag{6.7}$$

The form factors axioms (3.5) for the operator $\mathcal{T}$ still hold, with μ the double index (α, i), $\hat{\mu} = (\alpha, i+1)$ and $\bar{\mu} = (\bar{\alpha}, i)$. If particles α, β in the original theory fuse to produce a bound state γ, then the S-matrix has a pole on the imaginary line of the physical sheet, say $iu^{\gamma}_{\alpha\beta}$ with $u^{\gamma}_{\alpha\beta} \in (0, \pi)$. Correspondingly, the form factors will possess extra poles with the requirements

$$\begin{aligned} -i \lim_{\varepsilon \to 0} \varepsilon F^{\mathcal{T}|(\alpha,i)(\beta,i)\mu_1 \cdots \mu_{n-1}}_{n+1}\left(\theta + \frac{iu^{\gamma}_{\alpha\beta}}{2} - \varepsilon, \theta - \frac{iu^{\gamma}_{\alpha\beta}}{2} + \varepsilon, \theta_1, \dots, \theta_{n-1}\right) \\ = \Gamma^{\gamma}_{\alpha\beta} F^{\mathcal{T}|(\gamma,i)\mu_1 \cdots \mu_{n-1}}_{n+1}(\theta_1, \dots, \theta_{n-1}), \end{aligned} \tag{6.8}$$

where the so-called three-point coupling is

$$(\Gamma^{\gamma}_{\alpha\beta})^2 = -i \lim_{\theta \to i u^{\gamma}_{\alpha\beta}} (\theta - i u^{\gamma}_{\alpha\beta}) S_{\alpha\beta}(\theta). \tag{6.9}$$

As usual, for finding solutions to these equations we must first construct minimal solutions of the two-particle form factor equations. They satisfy the equations

$$F^{\mathcal{T}|(\alpha,j)(\beta,k)}_{\min}(\theta,n) = F^{\mathcal{T}|(\beta,k)(\alpha,j)}_{\min}(-\theta,n) S_{(\alpha,j)(\beta,k)}(\theta) = F^{\mathcal{T}|(\beta,k)(\alpha,j+1)}_{\min}(2\pi i - \theta, n), \tag{6.10}$$

for all values of j, k, α and β. From the equations above it follows:

$$F^{\mathcal{T}|(\alpha,i)(\beta,i+k)}_{\min}(\theta,n) = F^{\mathcal{T}|(\alpha,j)(\beta,j+k)}_{\min}(\theta,n), \quad \forall i, j, k, \alpha, \beta, \tag{6.11}$$

$$F^{\mathcal{T}|(\alpha,1)(\beta,j)}_{\min}(\theta,n) = F^{\mathcal{T}|(\beta,1)(\alpha,1)}_{\min}(2\pi(j-1)i - \theta, n), \quad \forall \alpha, \beta, j \neq 1. \tag{6.12}$$

So, as before computing the minimal form factors of particles in the first sheet is sufficient to determine all minimal form factors of the theory. This minimal form factor must have no poles in the extended strip $\mathrm{Im}(\theta) \in [0, 2\pi n]$ and satisfies a similar type of equations as for the case with $n = 1$,

$$F^{\mathcal{T}|(\alpha,1)(\beta,1)}_{\min}(\theta,n) = F^{\mathcal{T}|(\beta,1)(\alpha,1)}_{\min}(-\theta,n) S_{\alpha\beta}(\theta) = F^{\mathcal{T}|(\beta,1)(\alpha,1)}_{\min}(-\theta + 2\pi n i, n). \tag{6.13}$$

From arguments completely analogous to those developed for the one particle case, provided that the S-matrix of the original theory admits an integral representation of the form

$$S_{\alpha\beta}(\theta) = \exp\left[\int_0^{\infty} \frac{dt}{t} g_{\alpha\beta}(t) \sinh\left(\frac{t\theta}{i\pi}\right)\right], \tag{6.14}$$

where $g_{\alpha\beta}(\theta)$ is a function which depends of the theory under consideration with the property $g_{\alpha\beta}(t) = g_{\beta\alpha}(t)$ (that is, parity invariance), the minimal form factor is given by

$$F^{\mathcal{T}|\alpha\beta}_{\min}(\theta,n) = \exp\left[\frac{1}{2}\int_0^{\infty} \frac{dt}{t\sinh(nt)} g_{\alpha\beta}(t) \cosh\left(t\left(n + \frac{i\theta}{\pi}\right)\right)\right]. \tag{6.15}$$

We are now in the position to obtain the full two-particle form factors by including appropriate poles. In contrast to the single particle spectrum case not all two-particle form factors will have poles. The kinematic and bound-state residue equations ensure that the only singularities occur for form factors of the type $F_2^{\mathcal{T}|(\alpha,i)(\bar{\alpha},j)}(\theta,n)$ for any values of i, j, α, or of the type $F_2^{\mathcal{T}|(\alpha,i)(\beta,j)}(\theta,n)$ for any value of α, β for which there is a non-zero three-point coupling $\Gamma^{\gamma}_{\alpha\beta}$. If $\Gamma^{\gamma}_{\alpha\bar{\alpha}} = 0$, then the pole structure of these form factors is analogous to the one found for the one particle case

$$F_2^{\mathcal{T}|(\alpha,1)(\bar{\alpha},1)}(\theta) = \frac{\langle \mathcal{T} \rangle \sin(\frac{\pi}{n})}{2n \sinh(\frac{i\pi-\theta}{2n}) \sinh(\frac{i\pi+\theta}{2n})} \frac{F^{\mathcal{T}|(\alpha,1)(\bar{\alpha},1)}_{\min}(\theta,n)}{F^{\mathcal{T}|(\alpha,1)(\bar{\alpha},1)}_{\min}(i\pi,n)}. \tag{6.16}$$

In the presence of bound states, more factors need to be multiplied in order to account for the bound-state poles:

$$F_2^{\mathcal{T}|(\alpha,1)(\beta,1)}(\theta) = \frac{(A + B\cosh(\frac{\theta}{n}) + C\cosh(\frac{2\theta}{n})) F^{\mathcal{T}|(\alpha,1)(\beta,1)}_{\min}(\theta,n)}{(\sinh(\frac{i\pi-\theta}{2n}) \sinh(\frac{i\pi+\theta}{2n}))^{\delta_{\alpha,\bar{\beta}}} \sinh(\frac{iu^{\gamma}_{\alpha\beta}-\theta}{2n}) \sinh(\frac{iu^{\gamma}_{\alpha\beta}+\theta}{2n})}, \tag{6.17}$$

where A, B, C are constants that depend on α, β, n. For all other form factors we have

$$F_2^{\mathcal{T}|(\alpha,1)(\beta,1)}(\theta) \propto F_{\min}^{\mathcal{T}|(\alpha,1)(\beta,1)}(\theta), \quad \text{for } \alpha \neq \bar{\beta}, \qquad \Gamma_{\alpha\beta}^{\gamma} = 0. \tag{6.18}$$

6.1 Computation of the Entropy

We can now proceed to the computation of the entropy along the lines described in the previous section. The two-point function in the two-particle approximation is

$$\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle = \langle \mathcal{T}\rangle^2 \left(1 + \frac{n}{8\pi^2} \sum_{\alpha,\beta=1}^{\ell} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\theta_1 d\theta_2 f_{\alpha,\beta}(\theta_{12}, n)\, e^{-r(m_\alpha \cosh\theta_1 + m_\beta \cosh\theta_2)}\right), \tag{6.19}$$

where m_α is the mass associated to particle type α. The new function $f_{\alpha,\beta}(\theta, n)$ is given by

$$\begin{aligned} \langle \mathcal{T}\rangle^2 f_{\alpha,\beta}(\theta, n) &= \sum_{i=1}^{n} |F_2^{\mathcal{T}|(\alpha,1)(\beta,i)}(\theta, n)|^2 \\ &= |F_2^{\mathcal{T}|(\alpha,1)(\beta,1)}(\theta, n)|^2 + \sum_{j=1}^{n-1} |F_2^{\mathcal{T}|(\alpha,1)(\beta,1)}(-\theta + 2\pi i j, n)|^2. \end{aligned} \tag{6.20}$$

For general integrable QFTs it is also possible that the twist fields admit non-vanishing form factors both for odd and even particle numbers. If that would be the case there would be an extra contribution to the two-point function (6.19) coming from the 1-particle form factor. However this would have no contribution to the entropy, since the 1-particle form factors have no singularities in θ and therefore the 1-particle form factor contribution will exactly vanish at $n = 1$.

For the computation of the entropy we have exploited the pole structure of the form factors. In fact, from our analysis it follows that only form factors containing poles will contribute to the final value of $\tilde{f}_{\alpha,\beta}(1)$, where $\tilde{f}_{\alpha,\beta}(n)$ is the natural analytic continuation of $f_{\alpha,\beta}(0, n)$ from $n = 2, 3, \ldots$ to $n \in [1, \infty)$. More precisely, our analysis showed that only when poles collide do we have a non-zero contribution to the analytically-continued sum above at $n = 1$ (recall that the first term vanishes at $n = 1$). Since $0 < u_{\alpha\beta}^{\gamma} < \pi$, only kinematic poles can collide at $n = 1$, and this only occurs when $\theta = 0$. Selecting out those terms that contain kinematic poles, we obtain the constraint that $\beta = \bar{\alpha}$ in (6.19) for $n \sim 1$, which allows us to perform one of the integrals as before:

$$\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \approx \langle \mathcal{T}\rangle^2 \left(1 + \frac{n}{4\pi^2} \sum_{\alpha=1}^{\ell} \int_{-\infty}^{\infty} d\theta\, \tilde{f}_{\alpha,\bar{\alpha}}(\theta, n) K_0(2rm\cosh(\theta/2))\right) \quad (n \sim 1), \tag{6.21}$$

for $\tilde{f}_{\alpha,\beta}(\theta, n)$ the natural analytic continuation of $f_{\alpha,\beta}(\theta, n)$. From here, the results of the previous section easily generalise to

$$\tilde{f}_{\alpha,\bar{\alpha}}(1) = \frac{1}{2}, \qquad \left(\frac{\partial}{\partial n}\tilde{f}_{\alpha,\bar{\alpha}}(\theta, n)\right)_{n=1} = \frac{\pi^2}{2}\delta(\theta) \tag{6.22}$$

and

$$S_A(rm) = -\frac{c}{3}\log(\epsilon m) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O(e^{-3rm_1}), \tag{6.23}$$

where m_1 is the smallest mass and we admitted the possibility of non-zero three-particle form factors. In particular, the leading exponential correction is given by the mass of the lightest particles, and the coefficient is proportional to the number of such lightest particles in the spectrum.

7 Conclusion

In this paper we have developed a program which, taking full advantage of quantum integrable model techniques, allows for the computation of the entanglement entropy of a connected region of a quantum one-dimensional integrable system with respect to the remaining part of the system. The starting point of our computation is the well-known "replica-trick". Our program relies in the realization that starting with a quantum integrable model and constructing a new model consisting of n non-interacting copies of the original theory, a new local quantum field theory is obtained which naturally possesses $\mathbb{Z}_n$ symmetry. Associated to this symmetry two twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ exist, whose two-point function is directly related to the entropy of the system. Through the "replica-trick" mentioned above, the entropy is the derivative with respect to n of the two-point function of the twist fields, evaluated at $n = 1$.

Since the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are local fields of the n-copy theory, their two-point function can be computed by exploiting the form factor program for integrable models. More precisely, it can be expressed as a sum for different particle numbers over products of the form factors of the two fields involved. This gives a large-distance expansion: computing the two-point function in the two-particle approximation gives the leading behaviour at large distances. This expansion is in fact expected to converge rapidly, and this leading behaviour is often enough to describe the two-point function up to relatively small distances.

The behaviour of the entropy as a function of the distance was already well-known for very short and very large distances. For $r \ll m^{-1}$ (m being the mass of the lightest particle) that behaviour is determined by the underlying CFT which describes the integrable QFT in the ultraviolet limit. Thus, the entropy can be computed explicitly by using CFT techniques [18, 19]. At large separations $r \gg m^{-1}$ (in the infrared limit) the entropy is known to saturate to a constant value. The main result of our work has been to evaluate the first correction to the entropy in the infrared limit, providing therefore a description of the behaviour of the entropy in the intermediate region of values of rm. This correction is obtained from the two-particle contributions to the form factor expansion. In fact, the two-particle approximation provides both the saturation value of the entropy, coming from the disconnected part of the correlation function (the square of the vacuum expectation value of the twist fields), and the exact first corrections up to $O(e^{-3rm})$, coming from the two-particle contributions. We also computed the exact value of the saturation in the Ising model and showed it to be in good agreement with previous numerical results [11].

The most surprising result of our analysis has been to establish that the leading correction to the entropy at large rm is in fact a universal quantity, that is, it does not depend on the particular scattering matrix of the model we started with, but only on the spectrum of masses of the particles of the original theory. It is quite remarkable that the entropy should encode so explicitly crucial information about the theory both in its UV regime (the central charge) and in the IR regime (the number of light particles).

We have deduced this result from general arguments and checked it explicitly for the Ising and sinh-Gordon models. The mathematical reason for this "universal behaviour" is clear, as the result is directly related to the presence of kinematic poles in the two-particle form factors. Only form factors having such poles do contribute to the final result for the entropy, and their individual contributions turn out to be theory-independent. The presence of

bound state poles does not change our conclusions. However, we do not yet have a physical understanding of this result.

It would be interesting to compute higher order corrections to the entropy coming from states with higher particle numbers. In addition, an interesting application of our main result emerges, namely given a lattice model whose underlying QFT is unknown, a computation of the entropy could reveal the number of light particles of the theory by extracting the coefficient of the first exponential correction at large distances. Another interesting route is to compute the entanglement entropy of a disconnected region. This would involve higher-point correlation functions of the twist fields introduced here, and would be useful in verifying certain general properties of the entanglement entropy which follow from its interpretation as the number of "links" between the regions considered.

An additional result of our work has been to develop the form factor program for branch-point twist fields. As a consequence of the particular exchange relations between these fields and the fundamental fields of the theory, the form factor consistency equations for branch-point twist fields are different from those associated to standard local fields. In particular, the crossing and kinematic residue equations are modified. Since here we have only been concerned with the two-particle form factors of branch-point twist fields, an interesting open problem remains, namely to find closed solutions to the form factor equations for arbitrary or at least higher particle numbers. From the form factor equations we expect these solutions to be given in terms of elementary symmetric polynomials of the variables $e^{\theta_i/n}$. Also, it would be interesting to extend our analysis to non-diagonal theories, such as the sine-Gordon model.

In the Ising model, it is possible, for even n, to have an explicit representation of the branch-point twist fields using their relation to $U(1)$ twist fields in the free Dirac fermion model as shown in Appendix 2. This provides a way to explicitly evaluate Ising form factors, where many-particle form factors are obtained from two-particle form factors by the usual Wick's theorem. It also gives explicit representations of two-point functions in terms of the known Painlevé transcendent representations of the two-point functions of $U(1)$ twist fields. It would be interesting to study the Ising and Dirac models further from this viewpoint.

Finally, the notion of twist field as introduced in this work is new in the context of integrable QFT and can still be generalized further. It would be very interesting, for instance, to study twist fields in integrable models which possess non-abelian internal symmetries (such as $su(2)$). Developing the form factor program for such fields is a possible future line of research.

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Appendix 1: Form Factors of Branch-Point Twist Fields from Angular Quantisation in the Sinh-Gordon Model

It is possible to describe massive integrable quantum field theory using angular quantisation [34, 35]. Angular quantisation is obtained when the model is quantized on radial half-lines emanating from some fixed point in space. That is, the Hilbert space is some subspace of the space of field configurations on those radial half-lines, and time evolution is a rotation around that point. Let us consider the sinh-Gordon model, whose scattering matrix is given by (3.25). The angular quantisation of the model is obtained from free bosonic modes satisfying the canonical commutation relations

$$[\lambda_\nu, \lambda_{\nu'}] = f(\nu)\delta(\nu + \nu'), \tag{8.1}$$

with

$$f(\nu) = \frac{2\sinh\frac{\pi B\nu}{4}\sinh\frac{\pi(2-B)\nu}{4}}{\nu\cosh\frac{\pi\nu}{2}}, \tag{8.2}$$

along with the zero modes

$$[P, q] = -i. \tag{8.3}$$

The Hamiltonian of the system, generating rotation around the origin, is written in terms of the modes as

$$K = \int_0^\infty d\nu \frac{\nu}{f(\nu)}\lambda_{-\nu}\lambda_\nu. \tag{8.4}$$

One can easily calculate

$$[K, \lambda_\nu] = -\nu\lambda_\nu. \tag{8.5}$$

The angular Hilbert space, a Fock space over these modes, will be denoted by $\mathcal{F} = \oplus_p \mathcal{F}_p$. It is naturally decomposed into subspaces $\mathcal{F}_p$ with fixed eigenvalue p of the operator P (this determines the way the fundamental field behaves when it approaches the origin 0, which is, in angular quantisation, the far negative infinity of space).

The main advantage of this construction is that in some integrable models, an explicit embedding is known of the states in the usual quantisation scheme (with Hilbert space $\mathcal{H}$) into the space of operators acting on $\mathcal{F}$. This embedding goes as follows [35]. One first identifies vacuum expectation values in $\mathcal{H}$ with traces on $\mathcal{F}$, as a simple consequence of the change of quantisation scheme:

$$\langle \mathrm{vac}|A|\mathrm{vac}\rangle = \frac{\mathrm{Tr}(e^{-2\pi K}\iota(A))}{\mathrm{Tr}(e^{-2\pi K})} \equiv \langle\langle \iota(A)\rangle\rangle, \tag{8.6}$$

where on the left-hand sides A stands for an operator that is the representation on $\mathcal{H}$ of products of local fields, and on the right-hand side $\iota(A)$ stands for the representation on the angular-quantisation Hilbert space $\mathcal{F}$ of the same fields. Then, there are operators $Z(\theta)$ acting on $\mathcal{F}$ such that products of them correspond to asymptotic states of $\mathcal{H}$:

$$\langle \mathrm{vac}|A|\theta_1, \ldots, \theta_k\rangle = \langle\langle \iota(A) Z(\theta_1)\cdots Z(\theta_k)\rangle\rangle. \tag{8.7}$$

The operators $Z(\theta)$ are defined as follows. We have first

$$\Lambda^\eta(\theta) = :e^{-i\eta\int d\nu\lambda_\nu e^{i\nu(\theta - i\pi/2)}}: \tag{8.8}$$

for $\eta = \pm$, where the normal-ordering is with respect to the trace in (8.6), and we define

$$Z(\theta) = -iC\left[e^{\frac{i\pi P}{Q}}\Lambda^+(\theta + i\pi/2) - e^{-\frac{i\pi P}{Q}}\Lambda^-(\theta - i\pi/2)\right], \tag{8.9}$$

with

$$C = \frac{1}{\sqrt{\sin\frac{\pi B}{2}}}\exp\left[\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{Bt}{4}\sinh\frac{(2-B)t}{4}}{\sinh t\cosh\frac{t}{2}}\right]. \tag{8.10}$$

The representation $\iota(A)$ on $\mathcal{F}$ of any field at the origin that is local with respect to the fundamental field commutes with these operators; this constraint is expected to be sufficient

to fix the set of all such fields in angular quantisation (which excludes twist fields—they are discussed below).

For instance, form factors of exponential fields $e^{a\varphi}$ at the origin are obtained by choosing $\iota(A)$ to be a projector on $P = a$, times the vacuum expectation value $\langle e^{a\varphi}\rangle$. In particular, the identity operator is the projector on $P = 0$. Evaluating the traces is simple using

$$\langle\langle \lambda_\nu \lambda_{\nu'}\rangle\rangle = \frac{f(\nu)}{1-e^{-2\pi\nu}}\delta(\nu+\nu'), \tag{8.11}$$

obtained from cyclic properties of the trace. From the usual formula for evaluating averages of normal-ordered exponentials of free modes,

$$\langle\langle : e^{\int d\nu \lambda_\nu a(\nu)} : : e^{\int d\nu \lambda_\nu b(\nu)} :\rangle\rangle = e^{\int d\nu d\nu' a(\nu)b(\nu')\langle\langle \lambda_\nu \lambda_{\nu'}\rangle\rangle}, \tag{8.12}$$

we obtain

$$\begin{aligned}&\left\langle\!\!\left\langle \Lambda^{\eta_1}\left(\theta_1+\eta_1\frac{i\pi}{2}\right)\Lambda^{\eta_2}\left(\theta_2+\eta_2\frac{i\pi}{2}\right)\right\rangle\!\!\right\rangle \\ &\quad = \exp\left[-2\eta_1\eta_2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{Bt}{4}\sinh\frac{(2-B)t}{4}}{\sinh t\cosh\frac{t}{2}}\cosh t\left(1+i\frac{\theta_1-\theta_2}{\pi}-\frac{1}{2}(\eta_1-\eta_2)\right)\right].\end{aligned} \tag{8.13}$$

Two-particle form factors of exponential fields are then given by

$$\langle \mathrm{vac}|e^{a\varphi}|\theta_1,\theta_2\rangle = \langle e^{a\varphi}\rangle C^2 \sum_{\eta_{1,2}=\pm} e^{(\frac{i\pi a}{Q}-\frac{i\pi}{2})(\eta_1+\eta_2)}\left\langle\!\!\left\langle \Lambda^{\eta_1}\left(\theta_1+\eta_1\frac{i\pi}{2}\right)\Lambda^{\eta_2}\left(\theta_2+\eta_2\frac{i\pi}{2}\right)\right\rangle\!\!\right\rangle \tag{8.14}$$

and the constant C ensures that the value of the kinematic residue is correct.

Angular quantisation is also useful for studying twist fields. Since the "space" (or equal-time slices) of angular quantisation is just the half line, a twist field associated to a symmetry action σ is just the operator for the symmetry action itself on the angular Hilbert space:

$$\iota(\mathcal{T}_\sigma(0)) = \langle\mathcal{T}_\sigma\rangle\sigma_{\mathcal{F}} \tag{8.15}$$

and this also projects on the subspace with $P = 0$. Note that the symmetry action applied on a field, $[\sigma_{\mathcal{F}}, \iota(\mathcal{O}(x\neq 0))]$, inside the trace on the angular Hilbert space, does produce the symmetry transformation of the field $\mathcal{O}(x)$: since there is a commutator, there are two branches inserted at slightly different angular times, and the two branches cancel each other except around the point x. Also, these two branches can be deformed into one branch from $x=-\infty$ to $x=\infty$, so this is really the same object as the operator for the symmetry action on $\mathcal{H}$. When there is no commutator, that is when just $\sigma_{\mathcal{F}}$ is inserted inside the trace, then this does not produce symmetry transformation, rather it corresponds to the insertion of a local field, the twist field.

We now consider the n-copy sinh-Gordon model: the new angular quantisation Hilbert space is just $\oplus_{j=1}^n \mathcal{F}^{(n)}$, where all $\mathcal{F}^{(n)}$ are isomorphic to $\mathcal{F}$, and the asymptotic state operators are just $Z_j(\theta)$, for $j=1,2,\ldots,n$. They are built as before out of bosonic modes $\lambda_{j,\nu}$, $j=1,2,\ldots,n$ that commute for different values of j. The local twist field $\mathcal{T}=\mathcal{T}_\sigma$ associated to the symmetry $\sigma: j\leftrightarrow j+1 \bmod n$ is then just given by (8.15).

Form factors of $\mathcal{T}$ are then

$$\langle \mathrm{vac}|\mathcal{T}|\theta_1,\ldots,\theta_k\rangle_{\mu_1,\ldots,\mu_k} = \langle\mathcal{T}\rangle\tilde{C}^k\langle\langle Z_{\mu_1}(\theta_1)\cdots Z_{\mu_k}(\theta_k)\rangle\rangle_\sigma, \tag{8.16}$$

with

$$\langle\langle\cdots\rangle\rangle_\sigma = \frac{\langle\langle\sigma_{\mathcal{F}}\cdots\rangle\rangle}{\langle\langle\sigma_{\mathcal{F}}\rangle\rangle}. \tag{8.17}$$

We now take the operators $Z_\mu(\theta)$ to be composed of exponential operators normal-ordered with respect to this new trace, in order for the computation of the trace of their products to go as before. Such a change of normal-ordering, for exponential of free modes, just changes the normalisation, and the operators $\Lambda_\mu^\pm(\theta)$ all get the same normalisation change. The constant $\tilde{C}$ has been introduced in order to account for this, and can be fixed by the requirement that the correct value of the kinematic residue is obtained from the two-particle form factors evaluated below (it could also be determined directly by a Bogoliubov transformation).

Using the fact that the symmetry acts like $\sigma_{\mathcal{F}}\lambda_{j,\nu}\sigma_{\mathcal{F}}^{-1} = \lambda_{j+1,\nu}$ (with $j+n \equiv j$) and cyclic properties of the trace, we can evaluate the following averages:

$$\langle\langle\lambda_{j,\nu}\lambda_{1,\nu'}\rangle\rangle_\sigma = \frac{e^{-2(j-1)\pi\nu}f(\nu)}{1-e^{-2n\pi\nu}}\delta(\nu+\nu') \quad (j=1,2,\ldots,n). \tag{8.18}$$

Note that with $j=1$, the average is the one obtained by changing the angle around the origin to $2\pi n$. This is naturally expected: the branch points represented by these branch-point twist fields are just negative-curvature conical singularities with an angle of $2\pi n$. For $j>1$, the formula says that in evaluating form factors with two particles belonging to different copies, one can replace $\lambda_{j,\nu}$ by $e^{-2(j-1)\pi\nu}\lambda_{1,\nu}$. From formula (8.8), it is clear that this is equivalent to changing the particle l, of type j_l, to type 1, and shifting its rapidity θ_l to $\theta_l + 2\pi i(j_l-1)$, as long as the resulting integral representation for the form factor is convergent after this shift. This is so if the shift keeps the rapidity in the strip $\mathrm{Im}(\theta_1-\theta_2)\in[0,2\pi n]$, so that this shift is allowed if it is the first particle, $l=1$ with rapidity θ_1, that is changed. Hence, we find

$$F_2^{\mathcal{T}|j1}(\theta_1-\theta_2) = F_2^{\mathcal{T}|11}(\theta_1-\theta_2+2\pi i(j-1)), \tag{8.19}$$

which is in agreement with (3.11) (this equation holds for the whole form factor, not just the minimal form factor).

Hence it is sufficient to evaluate form factors involving particles on the same copy. The evaluation of traces involved in calculating these form factors goes as above, and we find

$$\begin{aligned}&\left\langle\!\!\left\langle \Lambda_1^{\eta_1}\left(\theta_1+\eta_1\frac{i\pi}{2}\right)\Lambda_1^{\eta_2}\left(\theta_2+\eta_2\frac{i\pi}{2}\right)\right\rangle\!\!\right\rangle_\sigma \\ &\quad = \exp\left[-2\eta_1\eta_2\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{Bt}{4}\sinh\frac{(2-B)t}{4}}{\sinh nt\cosh\frac{t}{2}}\cosh t\left(n+i\frac{\theta_1-\theta_2}{\pi}-\frac{1}{2}(\eta_1-\eta_2)\right)\right],\end{aligned} \tag{8.20}$$

which gives

$$\begin{aligned}\langle\mathrm{vac}|\mathcal{T}|\theta_1,\theta_2\rangle_{1,1} &= \langle\mathcal{T}\rangle(\tilde{C}C)^2\sum_{\eta_{1,2}=\pm}e^{-\frac{i\pi}{2}(\eta_1+\eta_2)}\left\langle\!\!\left\langle \Lambda_1^{\eta_1}\left(\theta_1+\eta_1\frac{i\pi}{2}\right)\Lambda_1^{\eta_2}\left(\theta_2+\eta_2\frac{i\pi}{2}\right)\right\rangle\!\!\right\rangle_\sigma \\ &= \langle\mathcal{T}\rangle(\tilde{C}C)^2\left(-2F_{\min}^{\mathcal{T}|11}(\theta)+\frac{1}{F_{\min}^{\mathcal{T}|11}(\theta+i\pi)}+\frac{1}{F_{\min}^{\mathcal{T}|11}(\theta-i\pi)}\right)\end{aligned} \tag{8.21}$$

(with $\theta = \theta_1 - \theta_2$), where $F_{\min}^{\mathcal{T}|11}(\theta)$ is given by (3.28). This can be seen to reproduce (3.18) if we choose

$$(\tilde{C}C)^2 = \frac{-iC\sqrt{\sin\frac{\pi B}{2}}}{2(\cos\frac{\pi B}{4} + \sin\frac{\pi B}{4} - 1)}, \tag{8.22}$$

using the properties

$$\frac{1}{F_{\min}^{\mathcal{T}|11}(\theta + i\pi)} = f(\theta)F_{\min}^{\mathcal{T}|11}(\theta), \qquad \frac{1}{F_{\min}^{\mathcal{T}|11}(\theta - i\pi)} = f(2\pi i n - \theta)F_{\min}^{\mathcal{T}|11}(\theta), \tag{8.23}$$

with

$$f(\theta) = \frac{\sin\frac{\pi}{2n}(\frac{B}{2} + 1 + \omega)\sin\frac{\pi}{2n}(-\frac{B}{2} + 2 + \omega)}{\sin\frac{\pi}{2n}(1 + \omega)\sin\frac{\pi}{2n}(2 + \omega)}, \tag{8.24}$$

and

$$\omega = n + \frac{i\theta}{\pi}. \tag{8.25}$$

Appendix 2: Vacuum Expectation Values of Branch-Point Twist Fields in the Ising Model

In this appendix, we use the relation between branch-point twist fields in the n-copy Ising model and the n independent $U(1)$ twist fields in the n-copy free massive Dirac theory, in order to deduce vacuum expectation values of the former from the known formulas for those of the latter. Note that this derivation is very similar to that employed in [21].

The symmetry $\sigma : i \mapsto i + 1 \bmod n$ associated to the branch-point twist field in the n-copy Ising model can certainly be diagonalised on the particle eigenstates. It would be natural to associate the resulting diagonal elements with elements of a $U(1)$ symmetry group, and to interpret the branch-point twist field as a product of the corresponding n independent $U(1)$ twist fields. This, of course, is unnatural in the n-copy Ising model, since there is no $U(1)$ symmetry in the individual copies. The trick is to further *double* the n-copy Ising model in order to make it into an n-copy Dirac theory. Denoting the fundamental real Majorana fermion fields by $\psi_{a,j}$, $\psi_{b,j}$, $\bar{\psi}_{a,j}$, $\bar{\psi}_{b,j}$ for $j = 1, 2, \ldots, n$ and fundamental Dirac spinor fermion field $\Psi_j = \begin{pmatrix} \Psi_{R,j} \\ \Psi_{L,j} \end{pmatrix}$, we have the relations

$$\Psi_{R,j} = \frac{1}{\sqrt{2}}(\psi_{a,j} + i\psi_{b,j}), \qquad \Psi_{L,j} = \frac{1}{\sqrt{2}}(\bar{\psi}_{b,j} - i\bar{\psi}_{a,j}). \tag{9.1}$$

Each copy of the Dirac fermion has a $U(1)$ symmetry: it is the symmetry under rotation between copies a and b of the Ising model, which occurs because the Ising model is quadratic. In the n-copy Dirac theory there is also an extra symmetry under $SU(n)$ transformations of the multiplet $\begin{pmatrix} \Psi_1 \\ \cdots \\ \Psi_n \end{pmatrix}$, again because the theory is quadratic. Diagonalising the branch-point twist field $\mathcal{T}_{\text{Dirac}}$ in the Dirac theory, which is done with a $SU(n)$ transformation as is seen below, it is possible to interpret the new basis as n new independent Dirac fermions. It is then possible that the branch-point twist field can be simply written as a product of n independent $U(1)$ twist fields acting on these independent fermions. This is useful for the

Ising model, because by uniqueness of the branch-point twist field (characterized by its basic branch-point property and its dimension), we have (recall that the central charge of the Dirac theory is 1)

$$\mathcal{T}_{\text{Dirac}} = \mathcal{T}_a \otimes \mathcal{T}_b, \tag{9.2}$$

where $\mathcal{T}_a$ and $\mathcal{T}_b$ are the branch-point twist fields in the copies a and b of the n-copy Ising model respectively.

In order to go into the details, we need to be more precise about the $SU(n)$ transformation that diagonalises $\mathcal{T}_{\text{Dirac}}$. It would be natural to expect that the Fourier transform $\Psi^{(q)} = \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i j q}\Psi_j$ is the appropriate transformation, but there is an important subtlety: the various copies of the Dirac fermions *commute* amongst each other, whereas to form new fermions with linear combinations of them, they need to *anti-commute*. This is clear from our choice of scattering matrix $S_{jk}(\theta) = 1$ amongst different copies $j \neq k$. Of course, from the viewpoint of the Hilbert space this is only a choice of basis, and it is possible to define a new basis with scattering matrix -1 amongst different copies. There are many ways of doing that, all with the same result. We will choose the following (here with a two-particle example for simplicity):

$$|\theta_1\theta_2\rangle^{\text{ac}}_{j_1,j_2} = \begin{cases} |\theta_1\theta_2\rangle_{j_1,j_2}, & j_1 \leq j_2, \\ -|\theta_1\theta_2\rangle_{j_1,j_2}, & j_1 > j_2. \end{cases} \tag{9.3}$$

Written using annihilation and creation operators for the basis with upper index ac, fermions of different copies anti-commute (and this is the only change). In the new basis, the symmetry σ acts as follows:

$$\sigma\Psi^{\text{ac}}_j = \begin{cases} \Psi^{\text{ac}}_{j+1}, & j = 1, \ldots, n-1, \\ -\Psi^{\text{ac}}_1, & j = n. \end{cases} \tag{9.4}$$

The extra minus sign is understood as follows: in a non-zero correlation function, there are an even number of fermion fields. Let us order them in increasing copy label. There will be, say, k fields with copy label n at the right. Applying the symmetry makes them into copy 1, and they are the only fields that break the order of increasing copy label. Bringing them back to the beginning gives a minus sign if they are of odd number. This is cancelled by the extra minus sign above.

Writing σ in matrix form, $\sum_{k=1}^{n}\sigma_{jk}\Psi_k = \sigma\Psi_j$, we find that the eigenvalues λ, $\sum_{k=1}^{n}\sigma_{jk}v_k = \lambda v_j$, satisfy

$$\lambda^n = -1 \Rightarrow \lambda = e^{\frac{i\pi p}{n}}, \quad p \text{ odd}. \tag{9.5}$$

The explicit $SU(n)$ transformation is

$$\Psi^{(p)} = \sum_{j=1}^{n}(e^{-i\pi pj/n} - e^{-i\pi p(1+j/n)})\Psi_j. \tag{9.6}$$

Hence, we may expect that, for n even,

$$\mathcal{T}_{\text{Dirac}} = \prod_{q=1}^{\frac{n}{2}} \mathcal{O}^{(2q-1)}_{\frac{2q-1}{2n}} \prod_{q=1}^{\frac{n}{2}} \mathcal{O}^{(-2q+1)}_{-\frac{2q-1}{2n}}, \tag{9.7}$$

where $\mathcal{O}_\alpha^{(p)}$ is the $U(1)$ twist field acting non-trivially only on the fermion fields $\Psi^{(p)}$, associated with the $U(1)$ element $e^{2\pi i\alpha}$, for $\alpha \in [0,1]$. The dimension of these twist fields is α^2, and the choice of the odd values of p above forming the product is dictated by the requirement of having the lowest total dimension. The dimension of the product of fields on the right hand side of (9.7) is

$$2\sum_{q=1}^{\frac{n}{2}}\left(\frac{2q-1}{2n}\right)^2 = \frac{1}{12}\left(n-\frac{1}{n}\right), \tag{9.8}$$

which agrees with the CFT prediction (2.8) for central charge $c=1$. By uniqueness of the branch-point twist field, the factorisation above must be the correct one. We checked in the case $n=2$ that our form factors agree with formula (9.7) along with the known form factors form $U(1)$-twist fields [43]. For n odd, similar calculations show that there are many lowest-dimension factorized operators, and that their dimension does not agree with the CFT prediction (it is higher). Hence, we expect that in this case, no such factorisation exists. This is not a problem, as we are interested in the analytic expression for the vacuum expectation value, which can be obtained from even values of n.

We can now read off the expectation values of $\mathcal{O}_\alpha$ from [44, 45]:

$$\langle\mathcal{O}_\alpha\rangle = \left(\frac{m}{2}\right)^{\alpha^2}\exp\int_0^\infty\frac{dt}{t}\left(\frac{\sinh^2\alpha t}{\sinh^2 t}-\alpha^2e^{-2t}\right) = \left(\frac{m}{2}\right)^{\alpha^2}\frac{1}{G(1-\alpha)G(1+\alpha)}, \tag{9.9}$$

which gives (5.14) (here $G(z)$ is Barnes' G-function).

Appendix 3: The Function $\tilde{f}(\theta, n)$

We now derive the full analytic continuation $\tilde{f}(\theta,n)$ in an integral representation. For simplicity, we will assume that $F_2^{\mathcal{T}|11}(\theta,n)$ is 0 at $\theta=0$ and that it vanishes exponentially as $|\theta|\to\infty$, as is observed in the Ising and the sinh-Gordon models. Let us write

$$\tilde{f}(\theta,n) = \sum_{j=0}^{n-1} s(\theta,j), \tag{10.1}$$

with $s(\theta,j) = F_2^{\mathcal{T}|11}(-\theta+2\pi ij,n)(F_2^{\mathcal{T}|11})^*(-\theta-2\pi ij,n)$. Consider now the following contour integral:

$$\int_C \pi\cot(\pi z)s(\theta,z)\frac{dz}{2\pi i}, \tag{10.2}$$

where C is the closed rectangular contour with vertices $(n-iL, n+iL, iL, -iL)$. Then, $s(\theta,z)$ decays exponentially as $\operatorname{Im} z\to\pm\infty$, and the contributions from the horizontal segments vanish as $L\to\infty$. Thanks to the quasi-periodicity of the integrand, $s(\theta,z+n) = S(\theta-2\pi iz)S(\theta+2\pi iz)s(\theta,z)$, the contributions from the vertical pieces amount to

$$\int_{-i\infty}^{i\infty}(S(\theta-2\pi iz)S(\theta+2\pi iz)-1)\pi\cot(\pi z)s(z)\frac{dz}{2\pi i}. \tag{10.3}$$

The residues from the simple poles at $z=j$, $j=1,\dots,n-1$ sum to $f(\theta,n)$. There are also simple poles of $s(z)$ at $z=\frac{1}{2}\pm\frac{\theta}{2\pi i}$ and $z=n-\frac{1}{2}\pm\frac{\theta}{2\pi i}$—the kinematic singularities. They

evaluate to (for real θ)

$$\tanh\left(\frac{\theta}{2}\right)\mathrm{Im}(F_2^{\mathcal{T}|11}(-2\theta+i\pi,n)-F_2^{\mathcal{T}|11}(-2\theta+2\pi in-i\pi,n)). \tag{10.4}$$

At $\theta=0$ this gives $-1/2$ using the kinematic residue equation, as it should. For any non-zero θ, it vanishes as $n\to 1$, like $(n-1)^2$. As an analytic function of θ, there are simple poles at $\theta=\pm i\pi(n-1)$, which are those that give the main contribution as $n\to 1$ for θ near to 0. These poles are

$$\frac{i\cot(\frac{\pi n}{2})}{2(\theta+i\pi(n-1))},\qquad -\frac{i\cot(\frac{\pi n}{2})}{2(\theta-i\pi(n-1))},$$

which indeed gives the behavior (5.5), with $\tilde{f}(1)=1/2$.

The full analytic continuation can now be written:

$$\begin{aligned}\tilde{f}(\theta,n)=&\tanh\left(\frac{\theta}{2}\right)\mathrm{Im}(F_2^{\mathcal{T}|11}(-2\theta+i\pi,n)-F_2^{\mathcal{T}|11}(-2\theta+2\pi in-i\pi,n))\\&-\frac{1}{4i\pi}\int_{-\infty}^{\infty}\coth\left(\frac{\beta}{2}\right)(S(\theta-\beta)-S(\theta+\beta))F_2^{\mathcal{T}|11}(-\theta+\beta,n)\\&\times(F_2^{\mathcal{T}|11}(\theta+\beta,n))^*d\beta.\end{aligned} \tag{10.5}$$

In particular, at $\theta=0$, this specializes to

$$\tilde{f}(n)=\frac{1}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}(S(-\beta))\coth\left(\frac{\beta}{2}\right)|F_2^{\mathcal{T}|11}(\beta,n)|^2d\beta. \tag{10.6}$$

Appendix 4: Computation of $\tilde{f}(\infty)$ for the Sinh-Gordon Model

In this appendix we present a detailed derivation of the equation of the line that provides the large n behaviour of the function $f(0,n)$ (and by analytic continuation that of $\tilde{f}(n)$) in the sinh-Gordon model. The computation can be carried out as follows: the limits of the various factors entering the two-particle form factor at $\theta=2\pi i(j-1)$ are given by

$$\begin{aligned}&\lim_{n\to\infty}F_{\min}^{\mathcal{T}|11}(2\pi i(j-1),n)=(j-1)\frac{\Gamma(j-\frac{1}{2}-\frac{B}{4})\Gamma(j-1+\frac{B}{4})}{\Gamma(j-\frac{B}{4})\Gamma(j-\frac{1}{2}+\frac{B}{4})},\quad \text{for } j\ll n,\\&\lim_{n\to\infty}F_{\min}^{\mathcal{T}|11}(i\pi,n)=\frac{1}{2}\frac{\Gamma(1-\frac{B}{4})\Gamma(\frac{1}{2}+\frac{B}{4})}{\Gamma(\frac{3}{2}-\frac{B}{4})\Gamma(1+\frac{B}{4})},\\&\lim_{n\to\infty}\frac{\sin(\frac{\pi}{n})}{2n\sinh(\frac{i\pi(2j-1)}{2n})\sinh(\frac{i\pi(3-2j)}{2n})}=\frac{2}{\pi(2j-1)(2j-3)},\quad \text{for } j\ll n,\end{aligned} \tag{11.1}$$

where for the first two functions, the limit can be easily evaluated by employing the integral representation (3.28) of the minimal form factor. Putting all factors together we obtain

$$\begin{aligned}&\lim_{n\to\infty}F_2^{\mathcal{T}|11}(2\pi i(j-1),n)\\&=\frac{4(j-1)}{\pi(2j-1)(2j-3)}\frac{\Gamma(j-\frac{1}{2}-\frac{B}{4})\Gamma(j-1+\frac{B}{4})\Gamma(\frac{3}{2}-\frac{B}{4})\Gamma(1+\frac{B}{4})}{\Gamma(j-\frac{B}{4})\Gamma(j-\frac{1}{2}+\frac{B}{4})\Gamma(1-\frac{B}{4})\Gamma(\frac{1}{2}+\frac{B}{4})},\end{aligned} \tag{11.2}$$

for $j \ll n$. In order to compute $f(0,\infty)$ we still need to perform the sum over j of the function above. Since the sum goes up to $j = n$ and in (11.2) we have assumed that $j \ll n$, it is not guaranteed that (11.2) will provide a good approximation for the whole range of values j. It turns out that (11.2) is the right function to consider for two reasons: first, computing $F_2^{\mathcal{T}|11}(2\pi i(j-1), n)$ numerically for various values of j and n one quickly realizes that it is only non-zero for values of j around 2 or around n; second, due to the periodicity of the form factor at $\theta = 0$, it turns out that the value of the function (11.2) does not change when $j \to n - j + 2$. Therefore, if we call the function (11.2) $k(j)$, we can replace $\sum_{j=2}^{n} k(j) \to 2\sum_{j=2}^{\infty} k(j)$ for $n \to \infty$. We then obtain

$$f(0,\infty) = \tilde{f}(\infty) = \frac{32}{\pi^2}\left[\frac{\Gamma(\frac{3}{2}-\frac{B}{4})\Gamma(1+\frac{B}{4})}{\Gamma(1-\frac{B}{4})\Gamma(\frac{1}{2}+\frac{B}{4})}\right]^2 A(B), \tag{11.3}$$

with

$$A(B) = \sum_{j=2}^{\infty} \frac{(j-1)^2}{(2j-1)^2(2j-3)^2}\left[\frac{\Gamma(j-\frac{1}{2}-\frac{B}{4})\Gamma(j-1+\frac{B}{4})}{\Gamma(j-\frac{B}{4})\Gamma(j-\frac{1}{2}+\frac{B}{4})}\right]^2. \tag{11.4}$$

Employing the definition of the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1), \tag{11.5}$$

we can rewrite the sum above as

$$A(B) = \left[\frac{\Gamma(\frac{3}{2}-\frac{B}{4})\Gamma(1+\frac{B}{4})}{\Gamma(2-\frac{B}{4})\Gamma(\frac{3}{2}+\frac{B}{4})}\right]^2 \sum_{j=2}^{\infty} \frac{(j-1)^2}{(2j-1)^2(2j-3)^2}\left[\frac{(\frac{3}{2}-\frac{B}{4})_{j-2}(1+\frac{B}{4})_{j-2}}{(2-\frac{B}{4})_{j-2}(\frac{3}{2}+\frac{B}{4})_{j-2}}\right]^2, \tag{11.6}$$

and carry it out analytically. The result is given in terms of generalized hypergeometric functions, which are defined as

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{11.7}$$

We find

$$\begin{aligned}
&\sum_{j=2}^{\infty} \frac{(j-1)^2}{(2j-1)^2(2j-3)^2}\left[\frac{(\frac{3}{2}-\frac{B}{4})_{j-2}(1+\frac{B}{4})_{j-2}}{(2-\frac{B}{4})_{j-2}(\frac{3}{2}+\frac{B}{4})_{j-2}}\right]^2 \\
&\quad = \frac{1}{9}\,{}_7F_6\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1, \alpha, \alpha, \beta, \beta \\ \frac{5}{2}, \frac{5}{2}, \kappa, \kappa, \sigma, \sigma \end{matrix}; 1\right] \\
&\qquad + \frac{1}{75}\left(\frac{\alpha\beta}{\kappa\sigma}\right)^2 {}_7F_6\left[\begin{matrix} \frac{3}{2}, \frac{3}{2}, 2, \alpha+1, \alpha+1, \beta+1, \beta+1 \\ \frac{7}{2}, \frac{7}{2}, \kappa+1, \kappa+1, \sigma+1, \sigma+1 \end{matrix}; 1\right] \\
&\qquad + \frac{2}{1225}\left(\frac{\alpha\beta}{\kappa\sigma}\right)^2 \frac{(8+B)^2(B-10)^2}{(10+B)^2(B-12)^2}\,{}_7F_6\left[\begin{matrix} \frac{5}{2}, \frac{5}{2}, 3, \alpha+2, \alpha+2, \beta+2, \beta+2 \\ \frac{9}{2}, \frac{9}{2}, \kappa+2, \kappa+2, \sigma+2, \sigma+2 \end{matrix}; 1\right],
\end{aligned} \tag{11.8}$$

with

$$\alpha = \frac{3}{2} - \frac{B}{4}, \qquad \beta = 1 + \frac{B}{4}, \qquad \kappa = 2 - \frac{B}{4}, \qquad \sigma = \frac{3}{2} + \frac{B}{4}. \tag{11.9}$$

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